Department of Mechanical Engineering

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LECTURE NOTES

Name of the Subject: Strength of Material

Semester: 3rd **Year:** 2nd

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1.0 **Load & Types of load:-**

A load may be defined as the combined effect of external forces acting on a body

(i) Loads can be classified as
- → Dead load
- → Live or fluctuating load
- → Inertia load
- → Centrifugal load.

(ii) Also it can be classified
- → Tensile load
- → Compressive load
- → Torsional or twisting load
- → Bending load
- → Shearing load.

(iii) Another classification of load according to pt. of application
- → Point load or concentrated load:-
  It is concentrated at a single pt.
- → Distributed load
  It is distributed over a cross section.
  - → Uniformly distributed load
  - → Uniformly varying load

2.0 **Stress:-**

When a body acted up on by some load it undergoes deformation i.e change in shape and size which increases gradually. During deformation, the material of the body resist the tendency of the load to deform the body, and when the load influence is taken over by the internal resistance of the mall of body, it becomes stable.

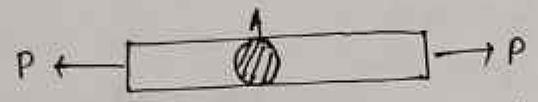
"The internal resistance offered by the body to meet with the load is called stress".

Stress can be considered either
(i) Total stress
(ii) Unit stress

(i) **Total stress:-** It is the total resistance to an external effect is called total stress. It is expressed in N, KN & MN.

(i) Unit Stress:-

The resistance developed per unit area is called unit stress. It is expressed in N/mm^2, MN/m^2 & KN/m^2.



$$\text{Stress} = \frac{\text{Resistance force}}{\text{Area}} = P/A$$

* $1\,MPa = 1\,MN/m^2$
$= 1 \times 10^6 N / (1000mm)^2$
$= \boxed{1\,N/mm^2}$

2.1 Types of stress:-

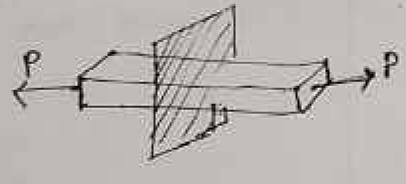
(i) Simple or direct stress
(a) Tension (b) compression (c) Shear

(ii) Indirect stress
(a) Bending (b) Twisting

(iii) Combined stress
combination of type-1 & type-2.

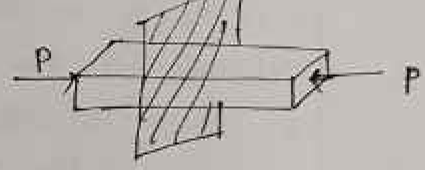
(i) (a) Tensile stress:-

The normal stress which causes extension on the member is called tensile stress. Here the applied force is pull. The force is acting perpendicular to the deformation plane.



(b) Compressive stress:-

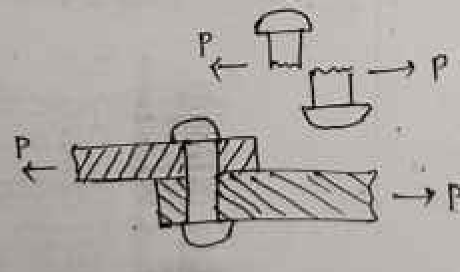
The normal stress which causes compression on the member is called compressive stress. Here the applied force is push. The force acting is perpendicular to the deformation plane.



* Both tensile & compressive stresses are called as normal stress as force is perpendicular to deformation plane.

(c) Shear stress:-

The force which is acting parallel to the cross-section of deformation is called shear force. The corresponding stress developed in the member is called shear stress.



3.0 Strain:-

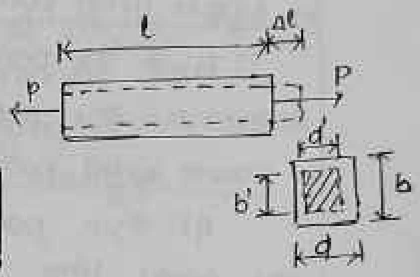
None of the materials are perfectly rigid. As we know when force is applied on rubber it extends. Similarly steel, cast iron, brass, concrete, etc. undergoes deformation when force applied on it. But the deformation is so small that it cannot be visible to naked eye. Extensometer, electric strain gauge are the instruments are used to measure the extension. The experiments shown that the bars extend at tensile load and shorten under compressive load.

Strain is defined as change in dimension to original dimension.

So during tensile load as the length increases the cross section decreases as total volume of the material is same and the vice-versa occurs at compressive load. So concept of two types of strain arises those are.

$$\boxed{\text{Linear strain } (e_t) = \frac{\text{change in length}}{\text{Original length}} = \frac{\Delta l}{l}}$$

$$\boxed{\text{Lateral strain } (e_d) = \frac{\text{change in crosssection}}{\text{original crosssection}} = \frac{\Delta A}{A}}$$

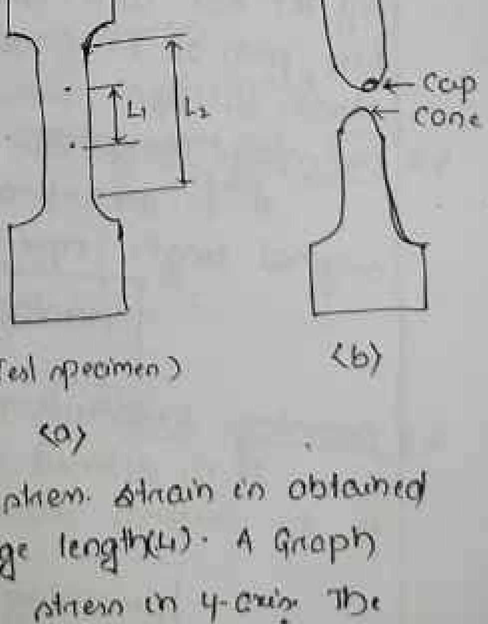


4.0 Stress-Strain Relation:-

The stress-strain relation of any material is obtained by conducting tensile test on standard specimen. Different matl. behave differently under tension and compression.

4.1 Behaviour on Tension (Mild steel specimen):-

As shown in fig a typical mild steel test specimen are gripped at universal testing machine (UTM) at its both ends. Extensometer is fitted to measure the extension over gauge length (L_1). Load is applied gradually at the ends and the load and the extension measured at regular intervals. After certain load extension increases at faster rate. Load is gradually increased till the specimen breaks (As shown in fig-b)



(Test specimen)

(a) (b)

Load divided by original crosssectional area is called as nominal stress or simply stress. Strain is obtained by dividing extensometer reading with gauge length(L_1). A Graph is prepared by taking strain in x-axis and stress in y-axis. The

(11)

graph is called as stress-strain diagram and the salient points on the graph are :-

→ (a) Limit of proportionality (A):-

It is the limiting value of stress up to which stress is directly proportional to strain.

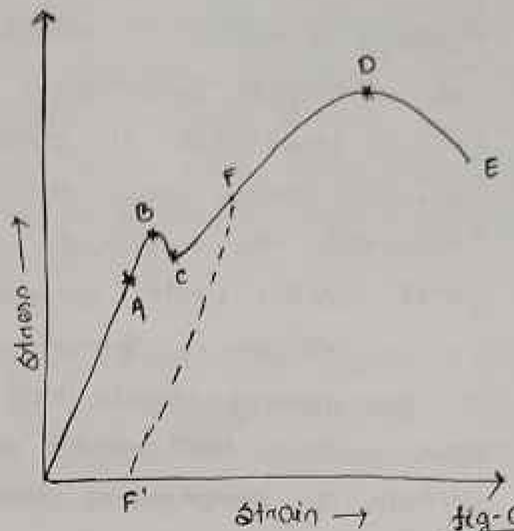


fig-c

→ Elastic limit :-

It is the limiting value of stress up to which if the load is removed it regains to its original size and the strain is said to be disappeared.

→ Upper yield point (B):-

It is the point at which the load starts reducing and the extension increases. The phenomenon is called yielding of material.

→ Lower yield point (C):-

At this point the stress remains same and the strain increases for some time.

→ Ultimate stress (D):-

This is the maximum stress the matl. can resist. At this stage cross-sectional area of the specimen starts reducing and is called neck formation.

→ Breaking point (E):-

The stress at which finally the specimen fails is called breaking point.

* If the matl. is unloaded within elastic limit the original length is regained. i.e the curve follows the same path reversely as shown in fig (c) but after the elastic limit if unloaded it follows a straight line path of FF'. Thus if it is loaded after elastic limit a permanent strain is left.

4.2 percentage Elongation:-

It is defined as the ratio of the final extension at rupture to original length expressed, as percentage.

$$\boxed{\text{percentage Elongation} = \frac{L' - L}{L} \times 100}$$

L' = Final length
L = Original length

4.3 percentage Reduction in area:-

It is defined as the ratio of maximum change in cross sectional area to the original cross-sectional area expressed as percentage.

$$\boxed{\text{percentage Reduction} = \frac{A - A'}{A} \times 100}$$

A' = Final Area
A = Original Area

4.4 Behaviour of material under compression:-

An there is a chance of buckling (laterally bending) of long specimen, for compression test short specimens are used. This test involves measurement of smaller changes in length.

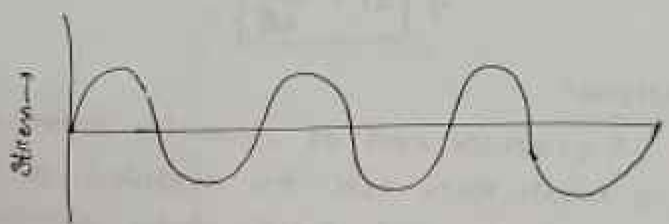
→ In case of ductile material stress strain curve follows exactly same path as in tensile test up to and even slightly beyond yield point. For larger values of curve diverges. In this case there will be no neck formation.

→ For most brittle materials ultimate compressive stress in compression is much larger than in tension. It is because of flaws and cracks present in brittle materials which weaken the matl. in tension but will not affect the strength in compression.

4.5. Behaviour of material under repeated loadings:-

when a material is subjected to reversal of stress that is from tensile to compressive and compressive to tensile then it is said to be repeated loaded. This type of varying stress affect the strength of material and this effect is called fatigue.

eg:- Shaft loaded with pulley rotating at a particular RPM.



5. Factor of Safety:-

In real practice it is not possible to design a member permitting up to ultimate stress due to following reasons.

→ Reliability of matl. may not 100% due to metallurgical defect.

→ Loads taken by the designer are only estimated loads.

→ Occasionally some members may be overloaded.

→ Some ideal conditions are assumed during design.

So, the maximum stress to which any member is designed is much less than the ultimate stress and is called working stress. The ratio of ultimate stress to working stress is called as factor of safety.

$$\boxed{\text{Factor of safety} = \frac{\text{ultimate stress}}{\text{working stress}}}$$

The value of factor of safety for some matl are

Steel - 1.85
Concrete - 3
Timber - 4 to 6

Hooke's Law

Robert Hooke, an English mathematician conducted several experiments and concluded that,

"Stress is directly proportional to strain with in elastic limit."

This is called as Hooke's law.

$\boxed{\sigma \propto e}$ $\boxed{\text{Stress} \propto \text{Strain}}$

Hence $\sigma = Ee$

Here, E is the constant of proportionality and known as modulus of elasticity or young's modulus.

Now, Stress (σ) = F/A

Strain (e) = $\Delta l / L$

According to Hooke's law

$\sigma = Ee \Rightarrow E = \frac{\sigma}{e} = \frac{F/A}{\Delta l / L} = \frac{FL}{A \cdot \Delta l}$

$\Rightarrow \boxed{\Delta l = \frac{FL}{AE}}$

problem:-

A circular rod of diameter 20mm and 500 mm long is subjected to a tensile force 45 kN. The modulus of elasticity for steel may be taken as 200 kN/mm². Find stress, strain and elongation of the bar due to applied load.

Sol

Given data,

Load (F) = 45 kN = 45000 N

Diameter (d) = 20 mm ⇒ Area = $\pi/4\, d^2 = \pi/4 \times (20)^2 = 314.159\ mm^2$

Length (L) = 500 mm

young's modulus (E) = 200 kN/mm² = $200 \times 10^3\ N/mm^2$

→ Stress (σ) = $F/A = \frac{45000}{314.159} = \boxed{143.24\ N/mm^2}$

→ ~~Strain (e)~~ :

→ Elongation (Δl) = $\frac{F \cdot L}{AE} = \frac{45000 \times 500}{314.159 \times 200 \times 10^3} = \boxed{0.358\ mm}$

→ Strain (e) = $\frac{\Delta l}{L} = \frac{0.358}{500} = \boxed{0.000716}$

problem :-

Following data refer to a mild steel specimen tested in a laboratory.

→ Diameter of the specimen = 25mm

→ Length of the specimen = 300 mm

→ Ext^n under a load of 15KN = 0.045mm

→ Load at yield point = 127.65KN

→ Maximum load = 208.60 KN

→ Length of specimen after failure = 375mm

→ Neck diameter = 17.75mm

Determine (a) young's modulus (b) yield stress (c) Ultimate stress (d) %age of elongation (e) % age of reduction (f) safe stress adopting a FOS = 2.

⑧

problem:-

A hollow steel tube is to be used to carry an axial compressive load of 140 kN. The yield stress of steel is 250 N/mm². A factor of safety of 1.75 is used in the design. The following three classes of tubes of external diameter 101.6 mm are available.

Light = 3.65 mm, Medium = 4.05 mm, Heavy = 4.85 mm (thickness)

which section do you recommend?

Soln: yield stress (σ_y) = 250 N/mm²

Factor of safety = 1.75

so, permissible stress (σ) = $\frac{\sigma_y}{FOS}$ = 250/1.75 = 142.857 N/mm²

Load (F) = 140 kN = 140×10^3 N.

Stress (σ) = F/A ⇒ $142.857 = \frac{140 \times 10^3}{A}$

⇒ $A = \frac{140 \times 10^3}{142.857} = 980\ mm^2$

we know

area of steel tube = $\pi/4\,(D_o^2 - D_i^2) = 980$ [D_o: Outer diameter, D_i: Inner diameter]

⇒ $\pi/4\,[(101.6)^2 - d^2] = 980$

⇒ $d^2 = 9074.784$ ⇒ $d = 95.262$ mm

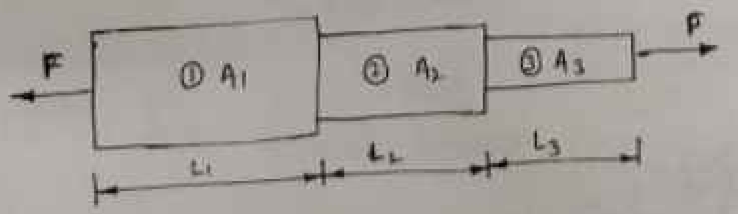
thickness (t) = $\frac{D-d}{2} = \frac{101.6 - 95.262}{2} = 3.169$ mm.

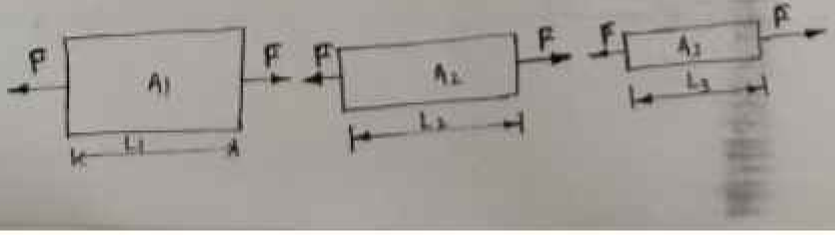
Hence, use of light section is recommended.

Bars with varying cross section:-

A bar with cross-sections varying in steps and subjected to axial load is as shown in fig. Let the lengths be l_1, l_2 & l_3 and the cross section be A_1, A_2 & A_3 respectively. 'E' be the young's modulus ~~the~~ and 'p' is the axial load applied.

To maintain the equilibrium the load acting is 'P' on the lines of action of the axial load. ~~It is based on principle of superposition~~





Let us divide the section in to three parts then load acting on each part is 'P' only as shown in fig

Now

Cross section	Stress	Strain	Elongation
1 →	$\sigma_1 = \frac{F}{A_1}$ →	$e_1 = \frac{\Delta l_1}{l_1}$ →	$\Delta l_1 = \frac{Fl_1}{A_1E}$
2 →	$\sigma_2 = \frac{F}{A_2}$ →	$e_2 = \frac{\Delta l_2}{l_2}$ →	$\Delta l_2 = \frac{Fl_2}{A_2E}$
3 →	$\sigma_3 = \frac{F}{A_3}$ →	$e_3 = \frac{\Delta l_3}{l_3}$ →	$\Delta l_3 = \frac{Fl_3}{A_3E}$

Here $\Delta l_1, \Delta l_2$ & Δl_3 are the elongations in small members.

Total Elongation $(\Delta L) = \Delta l_1 + \Delta l_2 + \Delta l_3$

$$= \frac{Fl_1}{A_1E} + \frac{Fl_2}{A_2E} + \frac{Fl_3}{A_3E} = \boxed{\frac{F}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right)}$$

problem:-

The bar as shown in fig is subjected to a load of 40 kN. then

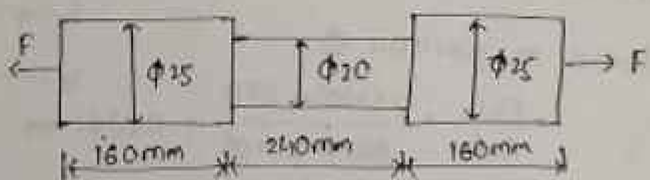
(a) Find the extension if $E = 210 \times 10^3 \, N/mm^2$

(b) Find the young's modulus value if total elongation is 0.325 mm.

Sol^n

Given data

$F = 40 \, kN = 40 \times 1000 \, N$



(a) young's modulus (E)

$= 210 \times 10^3 \, N/mm^2$

then $\Delta l_1 = \frac{Fl_1}{A_1E} = \frac{40 \times 10^3 \times 160}{\pi/4 (25)^2 \times 210 \times 10^3} = 0.0062$

$\Delta l_2 = \frac{Fl_2}{A_2E} = \frac{40 \times 10^3 \times 240}{\pi/4 \times (20)^2 \times 210 \times 10^3} = 0.145$

$\Delta l_3 = \frac{Fl_3}{A_3E} = \frac{40 \times 10^3 \times 160}{\pi/4 \times (25)^2 \times 210 \times 10^3} = 0.0062$

Total Extension $(\Delta L) = \Delta l_1 + \Delta l_2 + \Delta l_3 = 0.0062 + 0.145 + 0.0062$

$= \boxed{0.1574 \, mm}$

(b) Extension of portion ① $= \frac{FL_1}{A_1E} = \frac{40\times10^3\times160}{\pi/4\times25^2\times E}$

Extension of portion ② $= \frac{FL_2}{A_2E} = \frac{40\times10^3\times240}{\pi/4\times20^2\times E}$

Extension of portion ③ $= \frac{FL_3}{A_3E} = \frac{40\times10^3\times160}{\pi/4\times25^2\times E}$

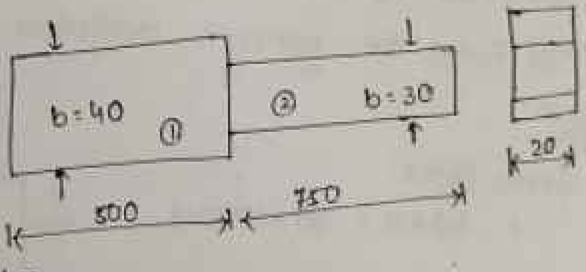
Total Extension $= \frac{40\times10^3}{E}\times\frac{4}{\pi}\left[\frac{160}{25^2}+\frac{240}{20^2}+\frac{160}{25^2}\right]$

$0.325 = \frac{40\times10^3}{E}\times\frac{4}{\pi}\times1.112$

$\Rightarrow E = \frac{40\times10^3\times4\times1.112}{0.325\times\pi} = \boxed{174257.52\ N/mm^2}$

problem:-

A stepped bar as shown in fig is made up of two different mat's the mat. ① has young's modulus $= 2\times10^5\ N/mm^2$, while that of mat. ② is $1\times10^5\ N/mm^2$. Find the extension of the bar under a pull of 25KN if both the portions are 20 mm thickness



sol^n $A_1 = 40\times20 = 800\ mm^2$

$A_2 = 30\times20 = 600\ mm^2$

Extension in portion ①

$\frac{FL_1}{A_1E_1} = \frac{25\times10^3\times500}{800\times2\times10^5} = 0.0781\ mm$

Extension in portion ②

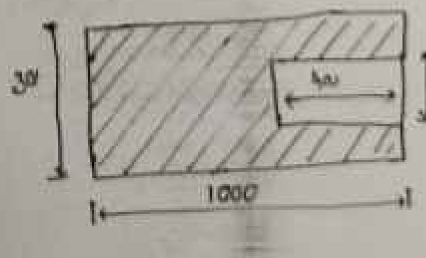
$\frac{FL_2}{A_2E_2} = \frac{25\times10^3\times750}{600\times1\times10^5} = 0.3125\ mm$

Total Extension of bar $= 0.0781 + 0.3125 = \boxed{0.3906\ mm}$

problem:-

A bar of length 1000 mm and diameter 30mm is centrally bored for 400 mm the bore dia being 10 mm. under the load 25KN. if the extension is 0.185mm. Find the modulus of elasticity of bar.

sol^n



Length of unbored area (L_1) =

$100 - 400 = 600mm$

Length of bored area (L_2) = 400mm

$A_1 = \pi/4 (30)^2 = 225\pi$

$A_2 = \pi/4 [30^2 - 10^2] = 200\pi$

$\Delta_1 = \frac{PL_1}{A_1 E}$

$\Delta_2 = \frac{PL_2}{A_2 E}$

$\Delta = \Delta_1 + \Delta_2 = \left[\frac{PL_1}{A_1 E} + \frac{PL_2}{A_2 E}\right]$

$= \frac{P}{E}\left[\frac{L_1}{A_1} + \frac{L_2}{A_2}\right]$

i.e. $0.185 = \frac{25 \times 10^3}{E}\left[\frac{600}{225\pi} + \frac{400}{200\pi}\right]$

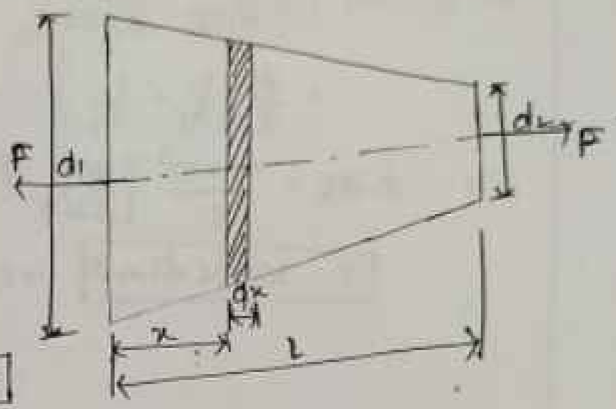
$\boxed{E = 200736 \text{ N/mm}^2}$

Extension of a tapering Rod:-

Consider a tapering bar having smaller diameter 'd_2', bigger diameter 'd_1' and length 'L' as shown in fig. when the cross section varies continiously an elemental length is considered and general expression for elongation of the elemental length derived. Then the general expression is integrated between boundary conditions of length to get total extension.

Consider an elemental length 'dx' of the bar at a distance of 'x' from the larger end.

d' = Diameter of the choosen crosssection = $d_1 - \left(\frac{d_1 - d_2}{L}\right)x$



Rate of change of dia $= \frac{d_1 - d_2}{L}$

at 'x' $= \frac{d_1 - d_2}{L} x$

$d' = d_1 - \left(\frac{d_1 - d_2}{L}\right)x$

$= \boxed{d_1 - Kx}$

Here $K = \frac{d_1 - d_2}{L}$

Cross sectional area $(A') = \frac{\pi}{4}(d')^2 = \frac{\pi}{4}(d_1 - Kx)^2$

Intensity of stress $(\sigma') = \frac{F}{A'} = \frac{F}{\frac{\pi}{4}(d_1 - Kx)^2} = \frac{4F}{\pi(d_1 - Kx)^2}$

Strain $(e') = \frac{\sigma'}{E} = \frac{4F}{E\pi(d_1 - Kx)^2}$

Extension in the elemental length $dx = e' \cdot dx$ = strain × length

$= \frac{4F}{E\pi(d_1 - Kx)^2} \cdot dx$

Total Extension along the length $(L) = \Delta l =$ $\qquad \int x^n = \frac{x^{n+1}}{n+1}$

$$\int_0^L \frac{4F}{E\pi(d_1 - Kx)^2} \cdot dx = \frac{4F}{E\pi}\int_0^L \frac{1}{(d_1 - Kx)^2}\, dx = \frac{4F}{E\pi}\left[-\frac{1}{d_1 - Kx} \cdot \frac{1}{-K}\right]_0^L$$

$$= \pm\frac{4F}{E\pi K}\left[\frac{1}{d_1 - Kx}\right]_0^L = \pm\frac{4F}{E\pi K}\left[\frac{1}{d_1 - KL} - \frac{1}{d_1 - K \times 0}\right]$$

$\cancel{\frac{4F}{E\pi K}\left[\frac{1}{d_1} - \frac{1}{d_1 - KL}\right]} = \pm\frac{4F}{\pi E K}\left[\frac{1}{d_1 - KL} - \frac{1}{d_1}\right]$

[~~−ve sign vanished due to ~~subtraction~~ constant from L to 0~~]

But we know that $K = \frac{d_1 - d_2}{L}$

So $\Delta L = \frac{4F}{\pi E K}\left[\frac{1}{d_1 - KL} - \frac{1}{d_1}\right] = \frac{4F \times L}{\pi E (d_1 - d_2)}\left[\frac{1}{d_1 - \frac{(d_1 - d_2) \times \cancel{L}}{\cancel{L}}} - \frac{1}{d_1}\right]$

$= \frac{4FL}{\pi E (d_1 - d_2)}\left[\frac{1}{\cancel{d_1} - \cancel{d_1} + d_2} - \frac{1}{d_1}\right]$

$= \frac{4FL}{\pi E (d_1 - d_2)}\left[\frac{1}{d_2} - \frac{1}{d_1}\right] = \frac{4FL}{\pi E \cancel{(d_1 - d_2)}}\left[\frac{(d_1 - d_2)}{d_1 d_2}\right]$

$$\Delta L = \boxed{\frac{4FL}{\pi E d_1 d_2}}$$

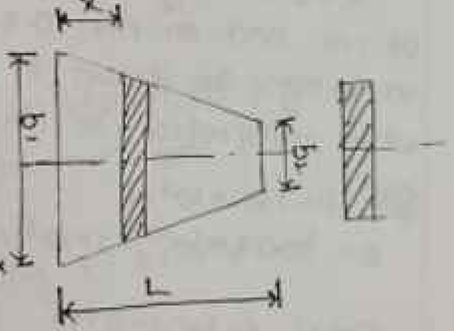
Note:-

when the rod is of uniform diameter $d_1 = d_2 = d$

$$\text{So } \Delta L = \frac{4FL}{\pi E d^2} = \frac{FL}{AE}$$

Rectangular rod:-

consider a small elemental area dA' at a distance x from bid end.



then,

Rate of change of breadth $= \frac{b_1 - b_2}{L}$

Breadth at distance 'x' $= \frac{b_1}{} - \left(\frac{b_1 - b_2}{L}\right)x$

$= b_1 - Kx \quad \left[K = \frac{b_1 - b_2}{L}\right]$

Area at that cross section $(A') = (b_1 - Kx)t$

Stress at A' $(\sigma') = \frac{F}{(b_1 - Kx)t}$

Strain at A' $(e') = \frac{\sigma'}{E} = \frac{F}{(b_1 - Kx)tE}$

Small elongation $(\Delta L')$ = strain × length (dx) $= \frac{F\,dx}{(b_1 - Kx)tE}$

Total Elongation $= \int_0^L \Delta L' = \int_0^L \frac{F\,dx}{(b_1 - Kx)tE} = \frac{F}{tE}\int \frac{dx}{b_1 - Kx}$ $\quad \boxed{y = b_1 - Kx,\ \frac{dy}{dx} = -K}$

$= \frac{F}{tE}\left(-\frac{1}{K}\right)\left[\log(b_1 - Kx)\right]_0^L$ $\quad \left[y = b_1 - Kx \Rightarrow \frac{dy}{dx} = -K \Rightarrow dx = \frac{dy}{-K};\ \int \frac{dx}{b_1 - Kx} = \int \frac{1}{y} \cdot \frac{dy}{-K} = -\frac{1}{K}\int \frac{dy}{y} = -\frac{1}{K}\log y \right]$

$$= \frac{F}{tEK}\left\{\left[-\cancel{\log(b_1 - K \times b)}\right] - \left[-\cancel{\log(b_1 - K0)}\right]\right\}$$

$$= \frac{F}{tEK}\left[\cancel{+\log b_1} - \left[\cancel{+\log b_1 - \frac{(b_1-b_2)}{K} \cdot K}\right]\right]$$

$$= \frac{F}{tEK}\left[\cdot \log b_1\right.$$

$$= \frac{F}{tEK}\left\{\left[-\log(b_1 - K \times L)\right] - \left[-\log(b_1 - K \times 0)\right]\right\}$$

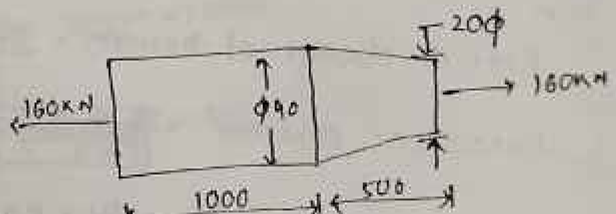
$$= \frac{F}{tEK}\left\{\left[-\log\left(b_1 - \frac{b_1 + b_2}{K} \times K\right)\right] - \left[\log b_1\right]\right\}$$

$$= \frac{F}{tEK}\left[\log b_1 - \log b_2\right]$$

$$= \frac{F}{tEK}\log\left(\frac{b_1}{b_2}\right) = \boxed{\frac{FL}{tE(b_1-b_2)}\log\left(\frac{b_1}{b_2}\right)}$$

problem:-

A 1.5 m long steel bar having uniform diameter of 40mm for a length of 1m and in next 0.5m its diameter gradually reduces from 40mm to 20 mm as shown in fig. Determine the elongation of this bar when subjected to an axial tensile load of 160 KN, Given $E = 200 GN/m^2$



Sol^n $P = 160 \times 10^3 N$

$E = 200 GN/m^2 = 2 \times 10^5 N/mm^2$

Total Extension =

Extension on uniform portion (Δ_1)

+

Extension at tapering portion (Δ_2)

$$= \frac{PL}{AE} + \frac{4PL}{E\pi d_1 d_2}$$

$$= \frac{160 \times 10^3 \times 1000}{\frac{\pi}{4} \times (40)^2 \times 2 \times 10^5} + \frac{4 \times 160 \times 10^3 \times 500}{2 \times 10^5 \times \pi \times 40 \times 20}$$

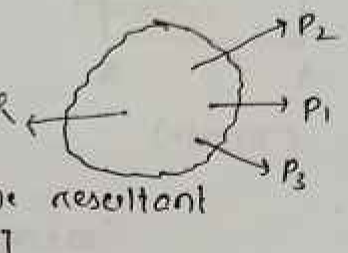
$= 0.6366 mm + 0.6366 mm$

$= \boxed{1.2732 mm}$

principle of superposition:-

It states that "the effect of a set of loads on a body is the same as the sum of effects caused by each load acting separately."

Mathematically.



Effect of 'R' = Effect of P_1 + Effect of P_2 + Effect of P_3 [If R is the resultant of P_1, P_2 & P_3]

This principle is valid for all linear (i.e stresses & resultant strains are linear) materials.

Bars subjected to varying loads:-

Bars may be subjected to different loads at different cross section. In such cases load acting in each portion may be found and the corresponding extensions on each portion determined. The addition of extensions of all portions of the bar gives its final extension.

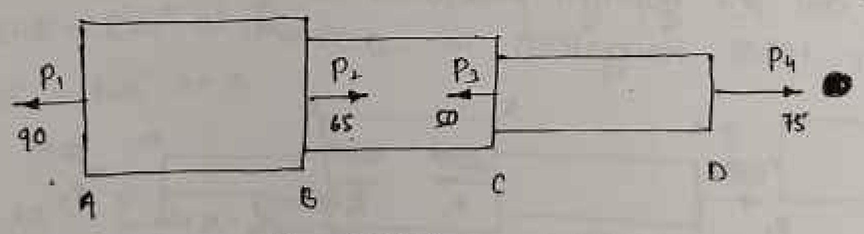
problem:-

The following details refer to a member as shown in fig.

AB = 40 mm × 40 mm, BC = 30 mm × 30 mm, CD = 20 mm × 20 mm.

AB = 500 mm, BC = 900 mm, CD = 800 mm.

Find the total extension by taking $E = 2 \times 10^5 \, N/mm^2$



$P_1 = 90$ KN, $P_2 = 65$ KN, $P_3 = 50$ KN.

Soln

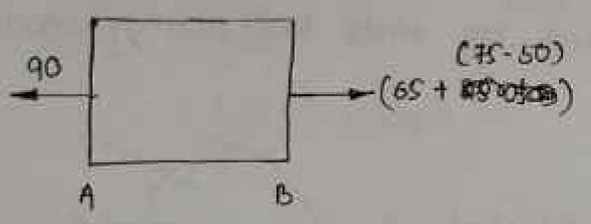
Since the body is in equilibrium

$P_1 + P_2 + P_3 + P_4 = 0$

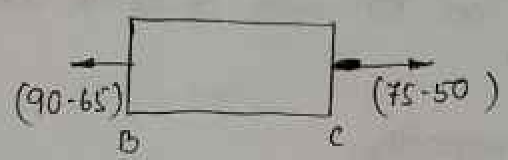
$\Rightarrow 90 - 65 + 50 + P_4 = 0 \Rightarrow \boxed{P_4 = -75}$

Now drawing the FBD of individual portions

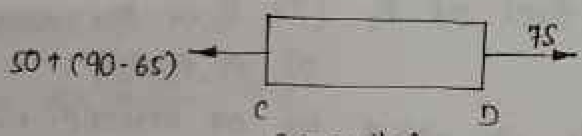
Assuming pt. A as fixed end



(Tensile)



(Tensile)



(Tensile)

<u>at AB (90 kN Tensile)</u>

$$\Delta l_1 = \frac{P_1 \times l_1}{A_1 E} = \frac{90\times10^3 \times 500}{40\times40 \times 2\times10^5} = 0.14\,mm$$

<u>At BC (25 kN Tensile)</u>

$$\Delta l_2 = \frac{P_2 l_2}{A_2 E} = \frac{25\times10^3 \times 900}{30\times30 \times 2\times10^5} = 0.125\,mm$$

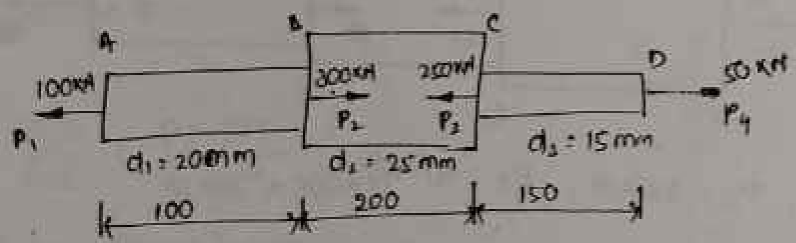
<u>At CD (75 kN Tensile)</u>

$$\Delta l_3 = \frac{P_3 l_3}{A_3 E} = \frac{75\times10^3\times 800}{20\times20\times2\times10^5} = 0.843\,m$$

Total Extension : $\Delta l_1 + \Delta l_2 + \Delta l_3 = 0.14 + 0.125 + 0.843 = \boxed{1.108\,mm}$

<u>problem</u>:-

Find out the young's modulus of the member as shown in fig when the total elongation is -0.04mm.

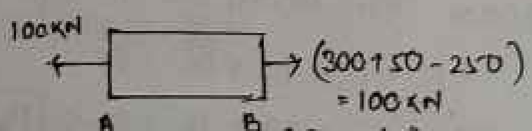


<u>Sol</u>n Considering equilibrium

$$P_1 + P_2 + P_3 + P_4 = 0$$

$$-100 + 300 - 250 + P_4 = 0$$

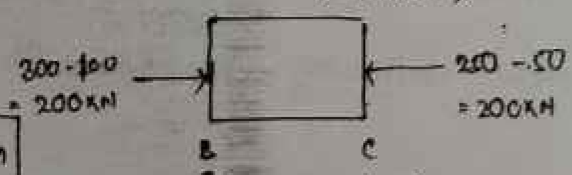
$$\Rightarrow \boxed{P_4 = 50\,kN}$$



(Tensile)

<u>Extn at AB</u>

$$\Delta l_1 = \frac{100\times10^3\times100}{\pi/4 \times 20^2 \times E} = \boxed{\frac{31847.13}{E}\,mm}$$

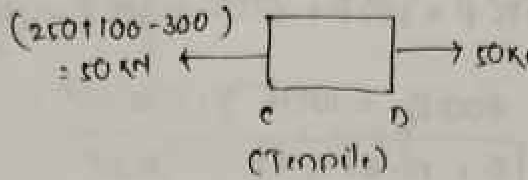


(Compressive)

Ext^n at BC

$\Delta l_2 = \dfrac{P_2 L_2}{A_2 E} = \dfrac{200\times10^3\times200}{\pi/4(25)^2\times E}$

$= \dfrac{81528.66}{E}$ (comp)



Ext^n at CD

$\Delta l_3 = \dfrac{P_3 L_3}{A_3 E} = \dfrac{50\times10^3\times150}{\pi/4\times(15)^2\times E} = \dfrac{42462.84}{E}$

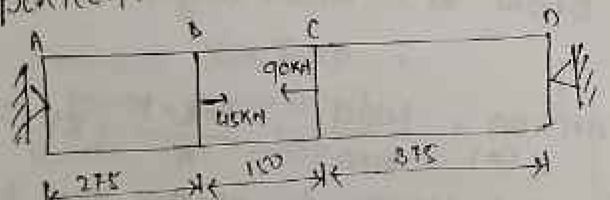
Total Extension $(\Delta l) = \Delta l_1 - \Delta l_2 + \Delta l_3$

$\Rightarrow \dfrac{31847.13}{E} - \dfrac{81528.66}{E} + \dfrac{42462.84}{E} = -0.04$

$\Rightarrow \dfrac{-7218.68}{-0.04} = E = \boxed{180467.12\ N/mm^2}$

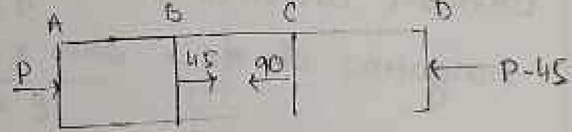
problem:-

A bar of 800 mm length is attached at A & B. Forces 45 kN & 90 kN acts as shown in fig. Take E = 200 GPa. Find the stresses & change in length in each portion.



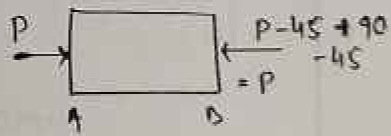
Sol^n

Consider the reaction at A is 'P'



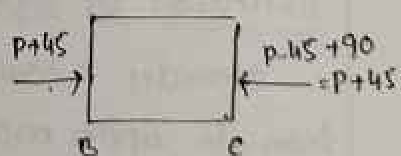
So $P - 90 + 45 = $ Rec^n at D

$\Rightarrow P - 45 = $ Rec^n at D



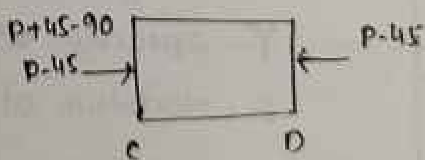
Now

Ext^n at AB $= \dfrac{P_1 L_1}{A_1 E} = \dfrac{P\times275}{\pi/4\times25^2\times200\times10^3}$



Ext^n at BC $= \dfrac{P_2 L_2}{A_2 E} = \dfrac{(P+45)\times150}{\pi/4\times25^2\times200\times10^3}$

Ext^n at CD $= \dfrac{P_3 L_3}{A_3 E} = \dfrac{(P-45)\times375}{\pi/4\times(25)^2\times200\times10^3}$



Total Ext^n = 0

$= \dfrac{P\times275 + (P+45)\times150 + (P-45)\,375}{\pi/4\times(25)^2\times200\times10^3} = 0$

$\Rightarrow 275P + 150P + 6750 + 375P - 16875 = 0$

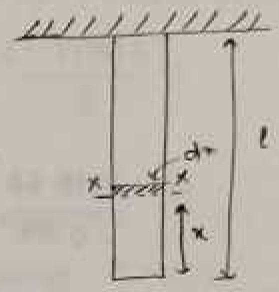
$\Rightarrow 800P = 10125$

$\Rightarrow \boxed{P = 12.65\ KN}$

* Find the stresses & extensions at each part by referring the FBD.

<u>Extension of a bar due to its own weight</u>:-

Consider a bar is hanging from its upper end.



Let d = Dia of bar
L = Length of bar.
γ = Specific weight = weight/~~unit length~~ volume
E = modulus of elasticity.

Now consider a section xx from x distance of lower end.

weight of rod below xx = Area × length × Specific weight
$= A \times x \times \gamma$

Stress at that section (σ) = $\frac{Load}{Area} = \frac{A \times x \times \gamma}{A} = x\gamma$

Consider an elemental length dx of the rod from xx.

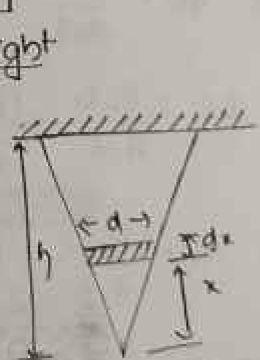
Elongation at dx = $\frac{Stress \times length}{E} = \frac{\sigma \times dx}{E} = \frac{x\gamma \cdot dx}{E}$

Total elongation of rod = $\int_0^L \frac{x\gamma dx}{E} = \frac{\gamma}{E}\int_0^L x\,dx$

$= \frac{\gamma}{E}\left[\frac{x^2}{2}\right]_0^L = \boxed{\frac{\gamma L^2}{2E}}$

<u>Extension of taper bar due to its own weight</u>

Consider a conical member hanging from its upper end.



L = Length of member
γ = Specific weight
E = Modulus of elasticity

Now consider a section x-x at a distance x from lower end.

weight of the rod below xx = vol of rod × sp. weight

$$= \frac{1}{3}\frac{\pi}{4}d^2 x \times \gamma$$

Stress at that section (σ) $= \frac{\text{Load}}{\text{Area}} = \frac{\frac{\pi}{12}d^2 x \gamma}{\frac{\pi}{4}d^2} = \frac{1}{3}x\gamma$

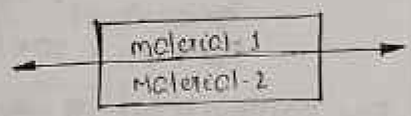
Extension on the member $= \frac{\sigma L}{E} = \frac{\frac{1}{3}x\gamma\, dx}{E}$

Total extension $= \int_0^L \frac{x\gamma\, dx}{3E} = \frac{\gamma}{3E}\int_0^L x\, dx = \frac{\gamma}{3E}\left[\frac{x^2}{2}\right]_0^L$

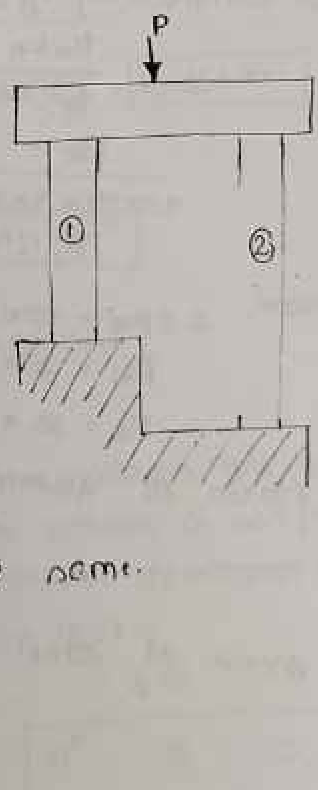
$$= \boxed{\frac{\gamma L^2}{6E}}$$

Compound Bars:

If a member consisting of two or more material subjected to axial forces. the members are called as compound members.



The different materials of the member may have same length or different lengths



Consider a member with two materials. Let the force developed due to applied loads on mat. ① and ② may be P_1 & P_2 respectively. Then according to static equilibrium condition along the axis of members

$$\boxed{P = P_1 + P_2}$$

Now by compatibility conditions

Deflection on both the members are same.

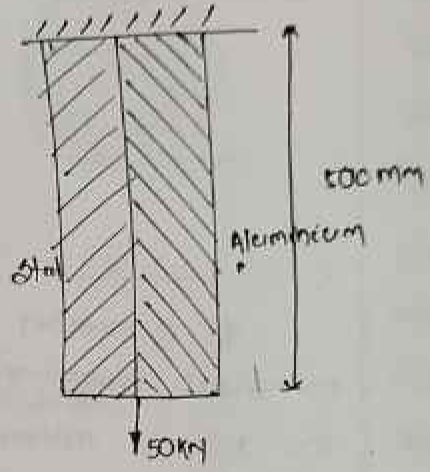
$$\Delta L_1 = \Delta L_2$$

$$\Rightarrow \boxed{\frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2}}$$

Problem:-

A compound bar of length 500 mm consists of a strip of Aluminium 50 mm wide × 20 mm thick and a strip of steel 50 mm wide × 15 mm thick rigidly joined at ends. If the bar is subjected to a load of 50 KN, find the stress developed in each material and the extension of the bar. Take elastic modulus of aluminium and steel as $1 \times 10^5\ N/mm^2$ and $2 \times 10^5\ N/mm^2$ respectively.

Sol^n



Pg · 34

$L = 500\,mm$

$P = 50KN = 50 \times 10^3 N$

$A_{al} = 20 \times 50 = 1000\ mm^2$

$A_{st} = 15 \times 50 = 750\ mm^2$

$E_{al} = 1 \times 10^5\ N/mm^2$

$E_{st} = 2 \times 10^5\ N/mm^2$

Let total load (P) = load on Aluminium (P_{al}) + Load on steel (P_{st})

$P_{al} + P_{st} = 50 \times 10^3 N$

For compatibility $\Delta_{al} = \Delta_{st}$

$$\frac{P_{al} L_{al}}{A_{al} E_{al}} = \frac{P_{st} L_{st}}{A_{st} E_{st}}$$

$$\frac{P_{al}}{20 \times 50 \times 1 \times 10^5} = \frac{P_{st}}{750 \times 2 \times 10^5} \quad [L_{al} = L_{st}]$$

$P_{st} = 1.5\, P_{al}$

Now $2.5 P_{al} = 50 \times 10^3 N$

$P_{al} = 20 \times 10^3 N$

$P_{st} = 30 \times 10^3 N$

Stress of Aluminium strip $= \frac{P_{al}}{A_{al}}$

$$= \frac{20 \times 10^3}{1000} = \boxed{20\ N/mm^2}$$

Stress of Steel strip $= \frac{P_{st}}{A_{st}}$

$$= \frac{30 \times 10^3}{750} = \boxed{40\ N/mm^2}$$

Strain $= (\Delta_{st}) = (\Delta_{al}) = \frac{30 \times 10^3 \times 500}{750 \times 2 \times 10^5}$

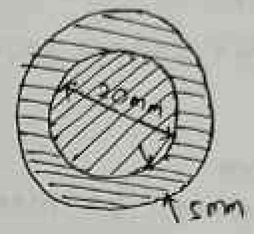
$$= \boxed{0.1\,mm}$$

Problem:-

A compound bar consists of a circular rod of steel of diameter 20mm rigidly fitted into a copper tube of internal diameter 20 mm and thickness 5mm as shown in fig. If the bar is subjected to a load of 100kN. Find the stresses developed in two materials if $E_s = 2\times10^5\ N/mm^2$, $E_c = 1.2\times10^5\ N/mm^2$.

Soln

$A_s = \pi/4 \times 20^2 = 100\pi\ mm^2$



External diameter of copper tube = $20 + 2\times5 = 30mm$

Area of copper tube $(A_c) = \frac{\pi}{4}\times(30^2 - 20^2) = 125\pi\ mm^2$

For static equilibrium

$$P = P_s + P_c$$

From compatibility condition

$$\Delta l_s = \Delta l_c$$

$$\Rightarrow \frac{P_s L}{A_s E_s} = \frac{P_c L}{A_c E_c} \Rightarrow \frac{P_s}{100\pi \times 2\times10^5} = \frac{P_c}{125\pi\times1.2\times10^5}$$

$$\Rightarrow \frac{P_s}{P_c} = \frac{100\times2}{125\times1.2} \Rightarrow \boxed{P_s = 1.33P_c}$$

Now,

$P_s + P_c = 100 \Rightarrow 1.33P_c + P_c = 100 \Rightarrow P_c = \frac{100}{2.33} = \boxed{42.857\ kN}$

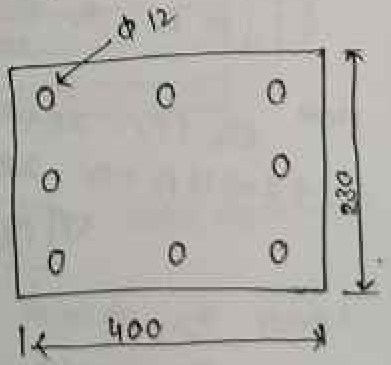
$P_s = 100 - 42.857 = \boxed{57.143\ kN}$

Stress on steel $(\sigma_s) = \frac{P_s}{A_s} = \frac{57.143\times10^3}{100\pi} = \boxed{181.89\ N/mm^2}$

Stress on copper $(\sigma_c) = \frac{P_c}{A_c} = \frac{42.857\times10^3}{100\pi} = \boxed{109.134\ N/mm^2}$

Problem:-

A reinforced concrete column of size 230mm × 400mm has 8 steel bars of 12mm diameter as shown in fig. If the column is subjected to an axial compression of 800kN, find the stresses developed in steel and concrete. Take modular ratio $E_s/E_c = 18.67$.



Soln

Area of steel $(A_s) = 8\times\frac{\pi}{4}\times d^2$

$= 8\times\frac{\pi}{4}\times(12)^2 = \boxed{904.779\ mm^2}$

Area of concrete $(A_c) = 230\times400 - A_s$

$= 230\times400 - 904.779$

$= 91095.221\ mm^2$

we know $P = P_s + P_c$

Now $\Delta L_c = \Delta L_s$

$\Rightarrow \frac{P_c L}{A_c E_c} = \frac{P_s L}{A_s E_s} \Rightarrow \frac{P_c}{91095.221 \times \cancel{E_c}} = \frac{P_s}{904.779 \times 18.67 \cancel{E_c}}$ $\left[\frac{E_s}{E_c} = 18.67\right]$

$\Rightarrow P_c = \frac{91095.221}{904.779 \times 18.67} P_s = \boxed{5.39 P_s}$

$P_s + P_c = 600\ KN$

$P_s + 5.39 P_s = 600\ KN \Rightarrow \boxed{P_s = 93.85\ KN}$ & $P_c = 5.39 P_s = 5.39 \times 93.81$

$\Rightarrow \boxed{P_c = 505.88\ KN}$

Hence

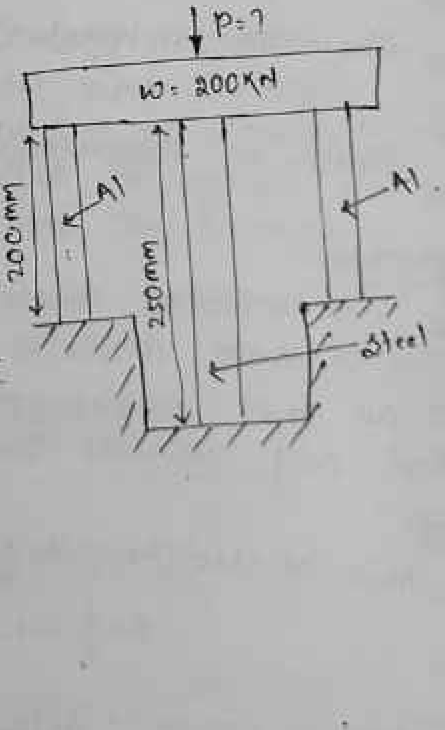
Stress on steel $= \frac{P_s}{A_s} = \frac{93.85 \times 10^3}{904.779} = \boxed{103.718\ N/mm^2}$

Stress on concrete $(\sigma_c) = \frac{P_c}{A_c} = \frac{505.88 \times 10^3}{91095.221} = \boxed{5.553\ N/mm^2}$

problem:-

Two pillars, two of aluminium and one of steel support a rigid platform of 200KN as shown in fig. If area of each aluminium pillar in 1000mm² and that of steel pillar in 800mm². Find the stresses developed in each pillar.

Take $E_a = 1 \times 10^5\ N/mm^2$ and $E_s = 2 \times 10^5\ N/mm^2$. what additional P can it take if working stresses are $65\ N/mm^2$ in aluminium and $150\ N/mm^2$ in steel?



Sol^n

(i) Due to 200KN load:-

Total force $= P_a + P_s + P_a = 200KN$

$\Rightarrow 2P_a + P_s = 200KN$

compatibility condition

$\Delta l_a = \Delta l_s$

$\frac{P_a L_a}{A_a E_a} = \frac{P_s L_s}{A_s E_s} \Rightarrow \frac{P_a \times 200}{1000 \times 1 \times 10^5} = \frac{P_s \times 250}{800 \times 2 \times 10^5}$

$\Rightarrow P_a = P_s \frac{250 \times 1000}{200 \times 800 \times 2} \Rightarrow \boxed{P_a = 0.78 P_s}$

Now $2P_a + P_s = 200KN$

$\Rightarrow 2 \times 0.78 P_s + P_s = 200KN \Rightarrow 2.56 P_s = 200KN$

$\Rightarrow \boxed{P_s = 78.04KN}$ $\boxed{P_a = 60.78KN}$

Hence the stresses are

Aluminium $(\sigma_a) = \frac{P_a}{A_a} = \frac{60.78 \times 10^3}{1000} = \boxed{60.78\ N/mm^2}$

Steel $(\sigma_s) = P_s/A_s = \frac{78.04\times10^3}{800} = \boxed{97.55\ N/mm^2}$

(ii) Additional load 'P' :-

Stress of Aluminium $(\sigma_a) = 65\ N/mm^2$

Load on Aluminium $(P_a) = \sigma_a \times A_a = 65\times1000 = 65000\ N$

Load on steel $(P_s) = \frac{P_a}{0.78} = \frac{65000}{0.78} = 83{,}333.33\ N$

Total load carrying capacity $= 2P_a + P_s$

$= 2\times 83333.33 + 65000$

$= 231666\ N = \boxed{231.66\ kN}$

Additional load $(P) = 231.66 - 200 = \boxed{31.66\ kN}$

(iii) Stress of steel $(\sigma_s) = 150\ N/mm^2$

Load on steel $(P_s) = \sigma_s \times A_s = 150\times800 = 1{,}20{,}000\ N$

Load on Aluminium $(P_a) = 0.78\,P_s = 0.78\times 1{,}20{,}000 = 93{,}600\ N$

Total load carrying capacity $= 2P_a + P_s$

$= 2\times93{,}600 + 1{,}20{,}000$

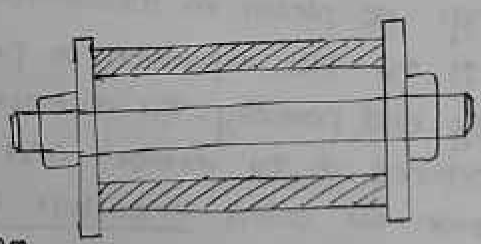
$= 3{,}07{,}200\ N = \boxed{307.2\ kN}$

Additional load $(P) = 307.2 - 200 = \boxed{107.2\ kN}$

problem:-

A steel bolt of 16 mm diameter passes centrally through a copper tube of internal diameter 20 mm and external diameter 30 mm. The length of the whole assembly is 500 mm. After tight fitting of the assembly, the nut is overtightened by quarter of a turn. What the stresses induced in bolt and tube, if pitch of nut is 2 mm. Take $E_s = 200\ GPa$, $E_c = 1.2\times10^5\ N/mm^2$.

Sol



P_s = Force acting on steel (Tensile)

P_c = Force acting on copper (compression)

$P_s + P_c = 0 \Rightarrow P_s = -P_c$

Let the force acting is 'P'.

Deflection $\Delta l = \frac{1}{4}$ pitch $= \frac{1}{4}\times 2 = 0.5\ mm$

As the bolt extended due to overtightening the tube shortened.

Net deflection

$\Delta l_c + \Delta l_s = \Delta l = 0.5\ mm.$

Area of steel (A_s): $\pi/_4 \times 16^2 = 201.062\ mm^2$

Area of copper (A_c): $\pi/_4 (30^2 - 20^2) = 392.699\ mm^2$

$$\Delta L = \Delta L_s + \Delta L_c = 0.5$$

$$= \frac{PL}{A_s E_s} + \frac{PL}{A_c E_c} \Rightarrow PL\left[\frac{1}{A_s E_s} + \frac{1}{A_c E_c}\right] = 0.5$$

$$\Rightarrow P \times 500 \left[\frac{1}{201.062 \times 2\times10^5} + \frac{1}{392.699 \times 1.2\times10^5}\right] = 0.5$$

$$\Rightarrow \boxed{P = 21697.331\,N}$$

$$\sigma_s = \frac{P}{A_s} = \frac{21697.33}{201.062} = \boxed{107.914\ N/mm^2}$$

$$\sigma_c = \frac{P}{A_c} = \frac{21697.33}{392.699} = \boxed{55.252\ N/mm^2}$$

Tempreture Stresses :-

when tempreture rises every material expands and contracts when tempreture falls. The change in length is directly proportional to the length of member and also tempreture change (Δt)

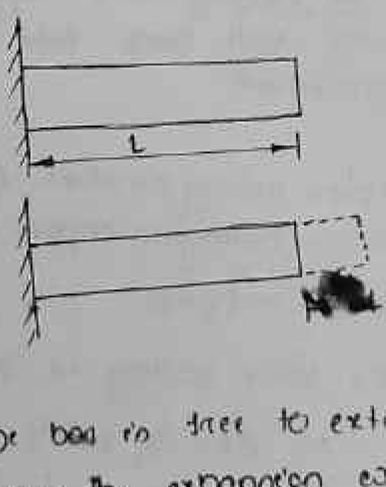
$$\Delta l \propto l\,\Delta t$$

$$\boxed{\Delta L = \alpha\,l\,\Delta t}$$

where α = co-efficient of thermal expansion and it is defined as change in unit length of material due to change in unit tempreture

If the changes due to tempreture are permitted freely no stresses develope in the member due to increase in tempreture the bar will extend by αtL due to this change no stress is introduced.

If the free expansion is prevented fully and partially, the stresses are introduced in the member and these stresses are called tempreture stresses.



consider a bar as shown in fig. If the bar is free to extend when tempreture is increased by 't' degrees the expansion would have been αtL. But if the extension is prevented completely the forces develope at supports. The effect of support force P is have a tendency to compress by αtL.

Hence

$\Delta = \alpha L \Delta t$

$\Rightarrow \frac{PL}{AE} = \alpha L \Delta t$

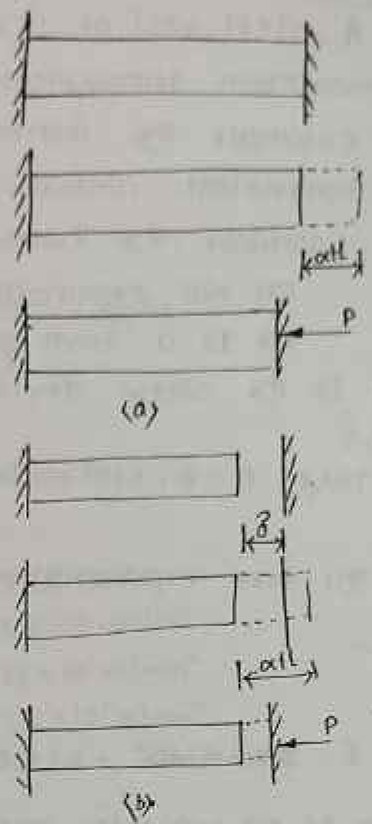
$\Rightarrow P/A = \alpha E \Delta t \Rightarrow \boxed{\sigma_t = \alpha t E}$

Here σ_t is the temperature stress and it is compressive in nature.

If the member is prevented partially as shown in fig-b then

$\Delta = \alpha t L - \delta$

$\Rightarrow \boxed{\frac{PL}{AE} = \alpha t L - \delta}$



Some standard values:-

Material	Coefficient of thermal Expansion
Steel →	12×10^{-6} /°c
copper →	17.5×10^{-6} /°c
Stainless steel →	18×10^{-6} /°c
Brass, Bronze →	19×10^{-6} /°c
Aluminium →	23×10^{-6} /°c

problem:-

A steel rail of 15m long and is laid at a temperature of 20°c. The maximum temperature expected is 40°c

→ Estimate the minimum gap between two rails to be left so that temperature stresses do not develop.

→ Calculate the thermal stresses developed in the rails if

(a) No expansion joint is provided.

(b) If a 2mm gap is provided for expansion.

→ If the stress developed is $25 MN/m^2$. what is the gap left betn the rails?

Take $E = 2.1 \times 10^5 MN/m^2$ and $\alpha = 15 \times 10^{-6}/°c$

Soln

→ The free expansion of the rails $= \alpha \Delta t L$

$= 15 \times 10^{-6} \times (40-20) \times 15 \times 1000$

$= \boxed{4.5mm}$

→ $E = \cancel{25 MN/m^2} = 2.1 \times 10^5 MN/m^2 = 2.1 \times 10^5 N/mm^2$

(a) If no expansion joint is provided, free expansion restricted is 4.5mm

i.e $\Delta L = 4.5 mm$

$\Rightarrow \frac{PL}{AE} = 4.5mm \Rightarrow \frac{P}{A} = \frac{4.5 \times E}{L} = \frac{4.5 \times 2.1 \times 10^5}{15 \times 1000} = \boxed{63 N/mm^2}$

(b) If 2mm gap is provided ($\delta = 2mm$)

$\Delta L = \alpha t L - \delta = 4.5 - 2 = 2.5mm$

$\frac{PL}{AE} = 2.5mm \Rightarrow \frac{P}{A} = \frac{2.5 \times E}{L} = \frac{2.5 \times 2.1 \times 10^5}{15 \times 1000} = \boxed{35 N/mm^2}$

→ If the stress developed is $25 MN/m^2 = 25 N/mm^2$ and δ is the gap

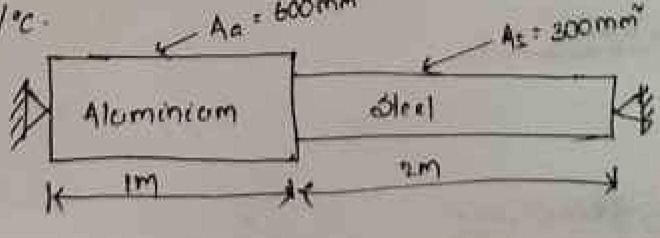
$\Delta L = \alpha t L - \delta = 4.5 - \delta$

$\frac{PL}{AE} = 4.5 - \delta \Rightarrow \frac{\sigma L}{E} = 4.5 - \delta \Rightarrow \frac{25 \times 15 \times 1000}{2.1 \times 10^5} = 4.5 - \delta$

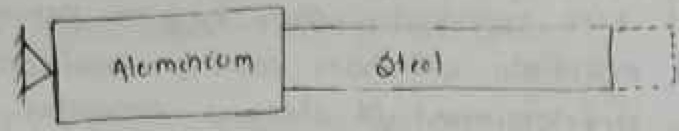
$\Rightarrow 1.78 = 4.5 - \delta \Rightarrow \delta = 4.5 - 1.78 = \boxed{2.71mm}$

problem:-

A composite bar is rigidly fitted at the supports A & B as shown in fig. Determine the reactions at the supports when temperature raises by 27°c. Take $E_a = 75 GN/m^2$, $E_s = 215 GN/m^2$, $\alpha_a = 12 \times 10^{-6}/°c$ and $\alpha_s = 15 \times 10^{-6}/°c$.



Soln



The free expansion would have been

$= \alpha_a \Delta t\, L_a + \alpha_s \Delta t\, L_s$

$= 12\times10^{-6} \times 27 \times 1000 + 15\times10^{-6} \times 27 \times 2000$

$= 0.324 + 0.81$

$= 1.134$ mm

Since this is prevented $\Delta l = 1.134$ mm

If P is support reaction the ΔL :

$$\frac{PL_a}{A_a E_a} + \frac{PL_s}{A_s E_s} \qquad \left[\begin{array}{l} E_a = 75\ GN/m^2 \\ = 75\times10^3\ MN/m^2 \\ = 75\times10^3\ N/mm^2 \\ E_s = 215\times10^3\ N/mm^2 \end{array}\right.$$

$$1.134 = P\left[\frac{1000}{600\times75\times10^3} + \frac{2000}{800\times200\times10^3}\right]$$

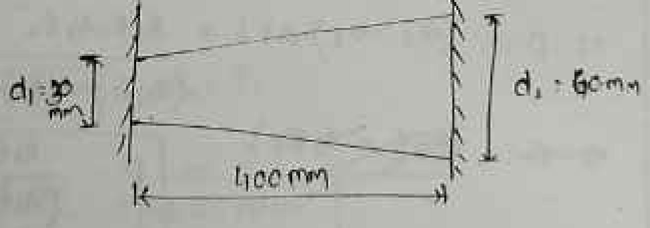
$1.134 = P \times 5.55\times10^{-5}$

$\boxed{P = 20412\ N}$

Example:-

A steel bar uniformly varying in diameter as shown in fig held between two unyielding support at room temperature. what is the maximum stress induced in the bar, if temperature raises by 30°c ? take $E_s = 2.1\times10^5$ N/mm², $\alpha_s = 15\times10^{-6}/°c$

Soln



Free expansion of the bar =

$\Delta L = \alpha \Delta t\, L$

$\Rightarrow 15\times10^{-6} \times 30 \times 400 = \boxed{0.18\ mm}$

If P is the force developed in restricting the expansion

$\Delta L = \frac{4PL}{\pi d_1 d_2 E} \Rightarrow \frac{4\times P\times 400}{\pi\times30\times60\times2.1\times10^5} = 0.18$

$\Rightarrow P = \frac{0.18\times\pi\times30\times60\times2.1\times10^5}{4\times400}$

$\Rightarrow \boxed{P = 135596.22\ N}$

Tempretare stresses in compound Bars:-

when tempretare rises the two materials of a compound bar experience different free expansions. since, they are prevented from separating, the two materials will have common position which is possible only by developement of stresses. consider a compound bar of length L. The rise in tempretare is Δt. α_1 & α_2 are the coefficient of thermal expansion and E_1 & E_2 are the elastic modulus.

Let $\alpha_1 > \alpha_2$

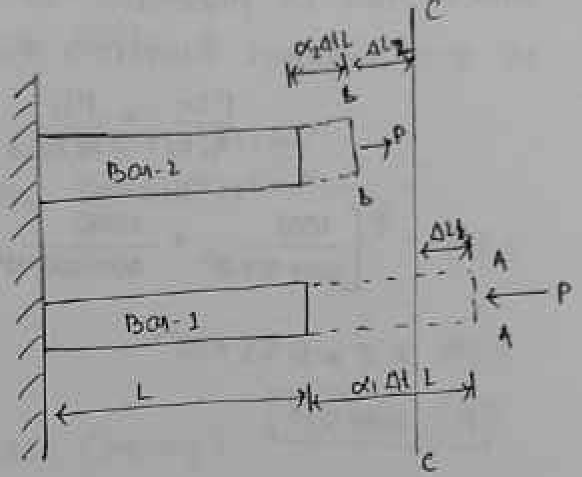
Free expansion of bar 1 $= \alpha_1 \Delta t L$

Free expansion of bar 2 $= \alpha_2 \Delta t L$

As $\alpha_1 > \alpha_2$ so $\alpha_1 \Delta t L > \alpha_2 \Delta t L$

Now a compressive stress will develope in bar-1 while tensile force will develope at bar-2

For equilibrium $P_1 = P_2 = P$



From the figure

$$\alpha_2 \Delta t L + \Delta L_2 = \alpha_1 \Delta t L - \Delta L_1$$

$$\Delta L_2 + \Delta L_1 = \alpha_1 \Delta t L - \alpha_2 \Delta t L$$

$$\Rightarrow \frac{PL}{A_1E_1} + \frac{PL}{A_2E_2} = (\alpha_1 - \alpha_2)\Delta t L$$

$$\Rightarrow PL\left[\frac{A_2E_2 + A_1E_1}{A_1E_1A_2E_2}\right] = (\alpha_1 - \alpha_2)\Delta t L$$

$$\Rightarrow P = (\alpha_1 - \alpha_2)\Delta t L \times \frac{A_1E_1A_2E_2}{(A_1E_1 + A_2E_2)L}$$

$$\Rightarrow \boxed{P = \frac{A_1E_1A_2E_2}{(A_1E_1 + A_2E_2)}(\alpha_1 - \alpha_2)\Delta t}$$

stresses in bar 1 $(\sigma_1) = P/A_1 = \frac{A_2E_1E_2}{(A_1E_1 + A_2E_2)}(\alpha_1 - \alpha_2)\Delta t$ (comp)

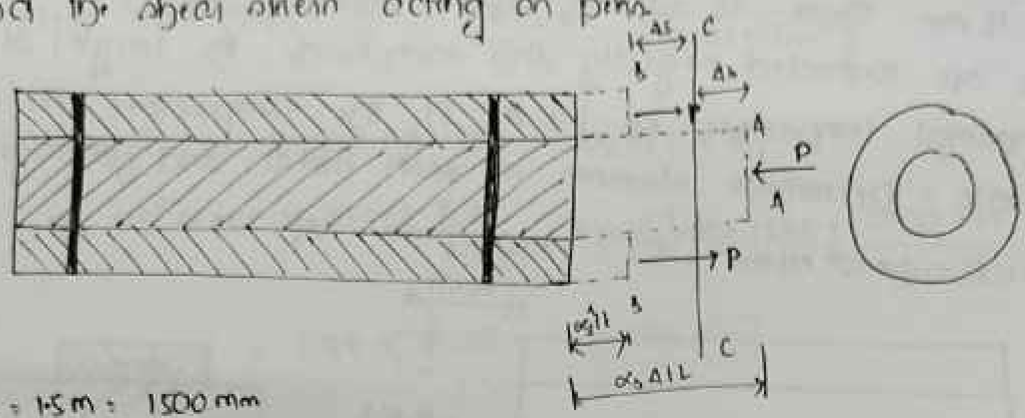
stress in bar 2 $(\sigma_2) = P/A_2 = \frac{A_1E_1E_2}{(A_1E_1 + A_2E_2)}(\alpha_1 - \alpha_2)\Delta t$ (Tensile)

problem:-

A bar of brass 30mm diameter is enclosed in a steel tube of 60mm diameter (external) and 30 mm internal diameter. The bar and the tube both initially 1.5 m long and are rigidly fastened at both ends by 20mm diameter pins. Find the stresses in two materials when temperature raise from 25°c to 75°c

Take $E_s = 200\ kN/mm^2$, $E_b = 100\ kN/mm^2$, $\alpha_s = 11.6\times10^{-6}/°c$, $\alpha_b = 18.7\times10^{-6}/°c$

Also find the shear stress acting on pins.



Now, $L = 1.5m = 1500mm$

$\Delta t = 75° - 25° = 50°c$

$A_b = \pi/4 \times (30)^2 = 706.5\ mm^2$

$A_s = \pi/4 \times (60^2 - 30^2) = 2119.5\ mm^2$

$E_s = 200\times10^3\ N/mm^2$

$E_b = 100\times10^3\ N/mm^2$

$P_b = P_s = P$

Now,

$\alpha_b \Delta t L - \Delta L_b = \alpha_s \Delta t L + \Delta L_s$

$\Rightarrow \Delta L_b + \Delta L_s = \alpha_b \Delta t L - \alpha_s \Delta t L = \Delta t L(\alpha_b - \alpha_s)$

$\Rightarrow \frac{PL}{A_b E_b} + \frac{PL}{A_s E_s} = 50 \times 1500 \times (18.7 - 11.6)\times10^{-6}$

$\Rightarrow PL\left[\frac{1}{706.5\times200\times10^3} + \frac{1}{2119.5\times100\times10^3}\right] = 0.5325$

$\Rightarrow P \times 1500 \left[7.07\times10^{-9} + 4.71\times10^{-9}\right] = 0.5325$

$\Rightarrow P = \frac{0.5325}{1500\times11.78\times10^{-9}} = \boxed{30135.82\ N} = P_b = P_s$

Stress in brass $(\sigma_b) = \frac{P_b}{A_b} = \frac{30135.82}{706.5} = \boxed{42.65\ N/mm^2}$

Stress in steel $(\sigma_s) = \frac{P_s}{A_s} = \frac{30135.82}{2119.5} = \boxed{14.21\ N/mm^2}$

Since when the pin will shear it will shear at two cross sections so it is the case of double shear.

$\tau = \frac{P}{2A_p} = \frac{30135.82}{2\times\pi/4\times(20)^2} = \boxed{47.98\ N/mm^2}$

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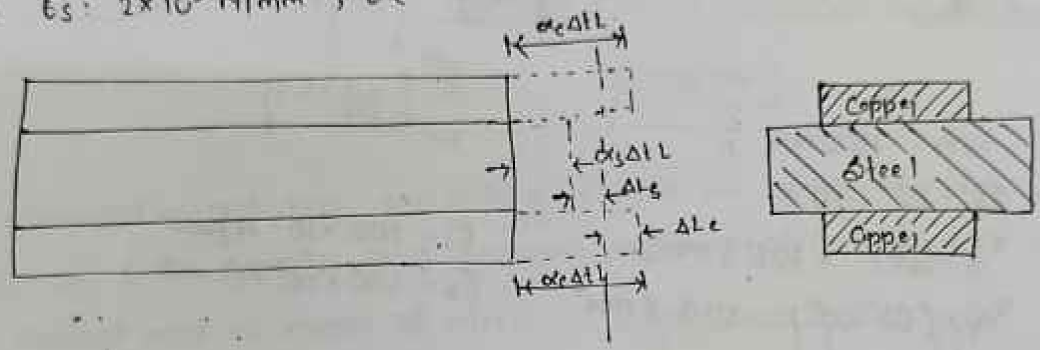
problem:-

A compound bar is made of a central steel plate 70 mm wide and 15 mm thick to which copper plates of 45 mm wide and 8 mm thick are connected rigidly on each side. The length of the bar at normal temperature is 1.2 m. If the temperature is raised by 80°c. Determine stresses in each metal and change in length. Take $E_s = 2\times10^5\ N/mm^2$, $E_c = 1\times10^5\ N/mm^2$, $\alpha_s = 12\times10^{-6}/°c$, $\alpha_c = 17\times10^{-6}/°c$



Sol^n

Given data,

$t = 80°C$, $L = 1.2m = 1200\ mm$, $E_s = 2\times10^5\ N/mm^2$, $E_c = 1\times10^5\ N/mm^2$,

$\alpha_s = 12\times10^{-6}/°c$, $\alpha_c = 17\times10^{-6}/°c$

$A_s = 70\times15 = 1050\ mm^2$, $A_c = 45\times8 = 360\ mm^2$

As $\alpha_c > \alpha_s$ so $\Delta L_c > \Delta L_s$ But plates are rigidly fastened. Hence to bring them to final position tensile force will act on steel and compressive force will act on copper.

Again $P_s = 2P_c$ [As two copper plates are there]

$$\alpha_c \Delta t L - \Delta L_c = \alpha_s \Delta t L + \Delta L_s$$

$$\Rightarrow \Delta L_c + \Delta L_s = \alpha_c \Delta t L - \alpha_s \Delta t L$$

$$\Rightarrow \frac{P_c L}{A_c E_c} + \frac{P_s L}{A_s E_s} = (\alpha_c - \alpha_s)\Delta t L$$

$$\Rightarrow \frac{P_c L}{A_c E_c} + \frac{2P_c L}{A_s E_s} = (\alpha_c - \alpha_s)\Delta t L \quad [\text{As } P_s = 2P_c]$$

$$\Rightarrow \left(\frac{P_c}{A_c E_c} + \frac{2P_c}{A_s E_s}\right) = (\alpha_c - \alpha_s)\Delta t$$

$$\Rightarrow P_c\left[\frac{1}{360\times1\times10^5} + \frac{2}{1050\times2\times10^5}\right] = (17-12)\times10^{-6}\times80$$

$$\Rightarrow P_c\left[2.77\times10^{-8} + 9.52\times10^{-9}\right] = 5\times10^{-6}\times80$$

$$\Rightarrow \boxed{P_c = 10746.9\ N}$$

$$\boxed{P_s = 2P_c = 21493.82\ N}$$

Stress in copper rod (σ_{cu}) = $\frac{P_c}{A_c} = \frac{10746.9}{360} = \boxed{29.85\ N/mm^2}$

Stress in steel rod (σ_s) : $\frac{P_s}{A_s} = \frac{21493.82}{1050} = \boxed{20.47\ N/mm^2}$

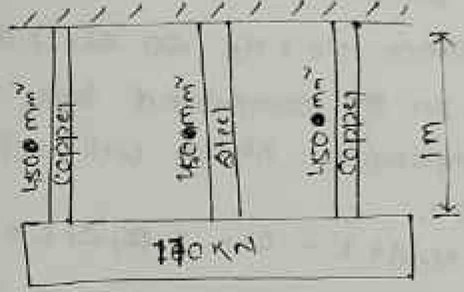
Change in length of compound bar = $\alpha_s \Delta t\, L + \Delta_s$

$$= \alpha_s \Delta t L + \frac{P_s L}{A_s E_s} = 12\times10^{-6} \times 80 \times 1500 + \frac{21493.82 \times 1500}{1050 \times 2\times10^5}$$

$$= 1.44 + 0.155 = \boxed{1.595\ mm}$$

Problem:-

A horizontal rigid bar weighing 170KN is hung by three vertical rods each of 0.9m length and $450mm^2$ cross-section as shown in fig. The central rod is made of steel and the outer rods are copper. If the temperature rise is 50°c estimate the load carried by each rod and by how much the load will descend



Take

$E_s = 200 GN/m^2$ $\quad E_c = 100 GN/m^2$

$\alpha_s = 1.2\times10^{-5}/°c$ $\quad \alpha_c = 1.8\times10^{-5}/°c$

what should be the temperature rise if the entire load 170KN is to be carried by steel alone?

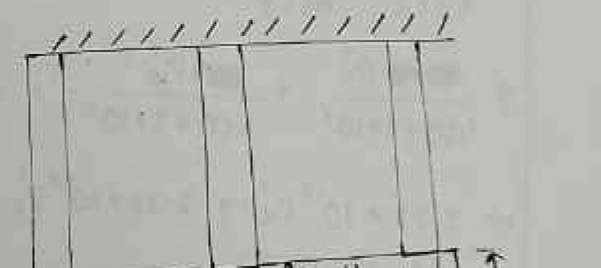
Sol:

Given data:

$E_s = 200 GN/m^2 = 200\times10^9 N/m^2$
$= 2\times10^5 N/mm^2$

$E_c = 100 GN/m^2 = 100\times10^9 N/m^2$
$= 1\times10^5 N/mm^2$

$L = 0.9m = 900mm$



(i) when only 170KN load will act:-

The system behaves as a compound bar. The initial change in length is due to weight 170KN.

$P_s + 2P_c = 170KN$

Extension in copper = Extension in steel

$\Delta L_c = \Delta L_s$

$$\frac{P_c L}{A_c E_c} = \frac{P_s L}{A_s E_s}$$

$$\Rightarrow \frac{P_c}{450 \times 1 \times 10^5} = \frac{P_s}{450 \times 2 \times 10^5} \Rightarrow P_s = 2P_c$$

Now

$$2P_c + P_s = 170\,kN$$

$$\Rightarrow 2P_c + 2P_c = 170\,kN$$

$$\Rightarrow P_c = 42.5\,kN$$

$$\Rightarrow P_s = 2P_c = 85\,kN$$

Extension due to loading $= \frac{P_s L}{A_s E_s} = \frac{85 \times 900 \times 1000}{450 \times 2 \times 10^5} = \boxed{0.85\ mm}$

(ii) Due to rise in temp by 50°C

Since $\alpha_c > \alpha_s$ so $\Delta L_c > \Delta L_s$

In the compound bar tensile stress will act on steel and compressive stress will act on copper.

Now,

$$\alpha_c \Delta t L - \Delta L_c = \alpha_s \Delta t L + \Delta L_s$$

$$\Rightarrow \Delta L_c + \Delta L_s = \alpha_c \Delta t L - \alpha_s \Delta t L$$

$$\Rightarrow \frac{P_c' L}{A_c E_c} + \frac{P_s' L}{A_s E_s} = (\alpha_c - \alpha_s)\Delta t L$$

$$\Rightarrow \frac{P_c'}{A_c E_c} + \frac{P_s'}{A_s E_s} = (\alpha_c - \alpha_s)\Delta t$$

$$\Rightarrow \frac{P_c'}{450 \times 1 \times 10^5} + \frac{2P_c'}{450 \times 2 \times 10^5} = (1.8 - 1.2) \times 10^{-5} \times 50$$

$$\Rightarrow 2.22 \times 10^{-8} P_c' + 2.22 \times 10^{-8} P_c' = 0.6 \times 10^{-5} \times 50$$

$$\Rightarrow P_c' = \boxed{6750N}$$

$$\Rightarrow P_s' = 2P_c' = \boxed{13500N}$$

Note:-
P_c' = stress on copper due to temp.
P_s' = stress on steel due to temp.

The amount of expansion due to temperature rise

$$\alpha_s \Delta t L + \Delta L_s = \alpha_c \Delta t L + \Delta L_c$$

$$\alpha_s \Delta t L + \frac{P_s' L}{A_s E_s} = 1.2 \times 10^{-5} \times 50 \times 900 + \frac{13500 \times 900}{500 \times 2 \times 10^5} = 0.54 + 0.1215 = \boxed{0.6615mm}$$

Total length by which the bar will descend =
Ext^n due to load + Ext^n due to temp rise

$$= 0.85 + 0.6615$$

$$= \boxed{1.5115\ mm}$$

Load carried by steel bar $= P_s + P_s' = 85000 + 13500 = \boxed{98{,}500\ N}$

Load carried by each copper rod $= P_c - P_c' = 42500 - 6750 = \boxed{35{,}750\ N}$

(iii) Rise in temperature if entire 170 KN load is to be carried out by steel rod $\Rightarrow P_s'' = 170\ KN = 170 \times 10^3\ N$

Extension due to load 170KN = Ext^n due to temp rise

$$\Rightarrow \frac{P_s'' \times L}{A_s \times E_s} = \alpha_s \Delta t\ L$$

$$\Rightarrow \frac{P_s''}{A_s E_s} = \alpha_s \Delta t \Rightarrow \frac{170 \times 10^3}{450 \times 2 \times 10^5} = 1.2 \times 10^{-5} \times \Delta t$$

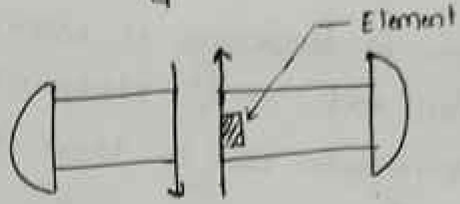
$$\Rightarrow \Delta t = \frac{170 \times 10^3}{450 \times 2 \times 10^5 \times 1.2 \times 10^{-5}} = \boxed{157.4\,°C}$$

Exercise problems:-

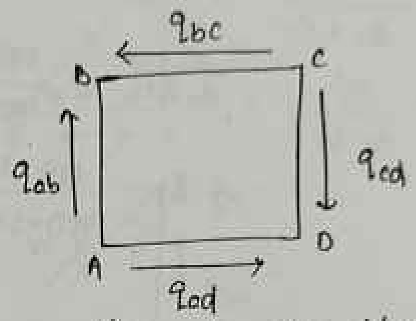
Simple shear:-

A material is said to be in a state of simple shear if it is subjected to only shearing stress. Shearing force tries to shear off the cross section of body.

Consider a bolt subjected to pure shear as shown in fig. Consider a small cross section of that element.



Let the intensity of shear stress be q_{ab} and thickness of element be 't'. Consider the equilibrium of element



(Element under shear)

vertical force on AB = $q_{ab} \times AB \times t$ —①

vertical force on CD = $q_{cd} \times CD \times t$ —②

Now for balancing

$$q_{ab} \times AB \times t = q_{cd} \times CD \times t \Rightarrow \boxed{q_{ab} = q_{cd} = q} \text{ —③}$$

Now q_{ab} & q_{cd} are equal in magnitude and opposite in direction also parallel to each other so they form one couple along moment arm AD.

i.e = $q_{ab} \times AB \times t \times AD$ —④

For balancing this couple another one couple will generate i.e q_{bc} & q_{cd}. Their direction is shown in fig (b).

Let $q_{bc} = q_{cd} = q'$ —⑤

So couple formed by these forces is

$$q \times AD \times t \times AB = q' \times AD \times t \times AB \text{ —⑤}$$

Equating eq^n ④ & ⑤

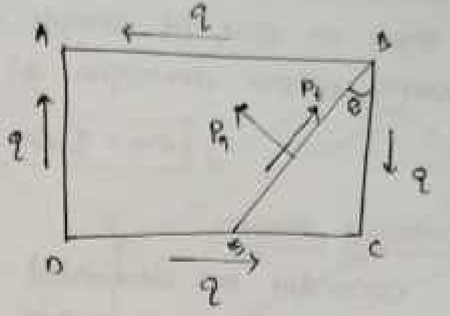
$$\boxed{q = q'}$$

Stresses in oblique section when element is in pure shear:-

Consider an elemental rectangular block ABCD. Let the thickness t = 1. The element offers a shear stress of intensity 'q' across the faces AB & CD and the faces AD & BC.

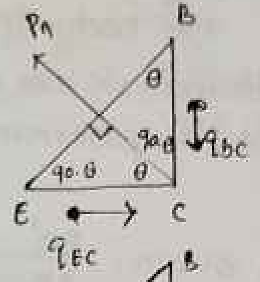
Consider a plane BE at an angle 'θ' with the face BC. Now the forces acting at the wedge section BCE (fig - d)

(i) Force $q \times BC$ along $BC \downarrow$
(ii) Force $q \times EC$ along $CE \rightarrow$
(iii) Force P_n normal to plane BE
(iv) Force P_t tangent to plane BE



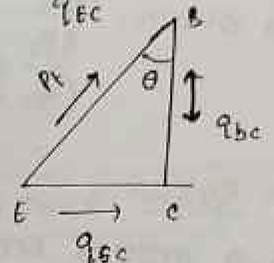
Resolving the forces normal to plane BE

$$P_n = q \times BC \times \sin\theta + q \times EC \times \cos\theta \quad \text{—— (6)}$$



Resolving the forces along the plane BE

$$P_t = q \times BC \times \cos\theta - q \times EC \times \sin\theta \quad \text{—— (7)}$$



Normal stress acting on plane BE

$$\sigma_n = \frac{P_n}{BE} = \frac{q \times BC \sin\theta}{BE} + \frac{q \times EC \cos\theta}{BE}$$

$$= q \times \cos\theta \cdot \sin\theta + q \times \sin\theta \cdot \cos\theta$$

$$= 2q \sin\theta \cdot \cos\theta = q \sin 2\theta \quad \text{—— (8)}$$

Tangential stress along the plane BE

$$\sigma_t = \frac{P_t}{BE} = \frac{q BC \cdot \cos\theta - q CE \cdot \sin\theta}{BE}$$

$$= q \cdot \cos\theta \cdot \cos\theta - q \sin\theta \cdot \sin\theta = q(\cos^2\theta - \sin^2\theta)$$

$$= q \cos 2\theta \quad \text{—— (9)}$$

Hence the normal and tangential stresses at plane BE

$$\sigma_n = q \sin 2\theta$$
$$\sigma_t = q \cos 2\theta$$

Note:-

→ Hence the plane carrying maximum normal stress the $\sin 2\theta$ value must be ± 1

$$\Rightarrow \sin 2\theta = \pm 1 \Rightarrow 2\theta = \pm 90^\circ \Rightarrow \boxed{\theta = \pm 45^\circ}$$

(+ for tensile & − for compressive)

→ Hence the plane where normal stress is maximum shear stress must be zero

→ For maximum value of shear stress the $\cos 2\theta$ value must be ± 1

$$\Rightarrow \cos 2\theta = \pm 1 \Rightarrow 2\theta = 0^\circ \text{ or } 180^\circ$$
$$\Rightarrow \boxed{\theta = 0^\circ \text{ or } 90^\circ}$$

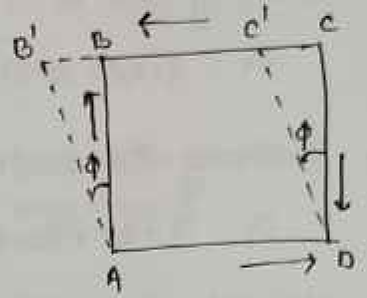
"Thus in case of simple shear tensile stress of same magnitude as shearing stress develops at 45° to shearing plane."

$$\boxed{\sigma_n = q \sin 2\theta = q \sin 2\times 45^\circ = q}$$

Shearing Strain:-

Consider an elemental cross section ABCD as shown in fig. Shearing stress has a tendency to distort the body to a position AB'C'D.

This deformation is expressed in terms of angular displacement and called shear strain.

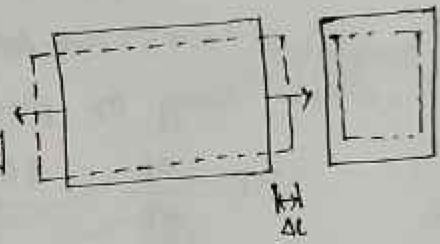


$$\text{Shear strain} = \frac{BB'}{AB} = \tan\phi = \phi$$

(As ϕ is very small)

Poisson's Ratio:-

When a material undergoes change in one dimension corresponding other dimension also changes. For example If due to tensile force the length of bar increases correspondingly its cross section decreases.



The ratio of change in length to original length is linear strain. Similarly the ratio of change in cross section to original cross section is lateral strain.

It is found that with in elastic limit there is a constant ratio between lateral strain to linear strain and that ratio is called as poisson's ratio. It is denoted by $1/r$ or μ. For most metals its value lies between 0.25 to 0.33.

$$\boxed{\mu = \frac{1}{r} = \frac{\text{lateral strain}}{\text{linear strain}}}$$

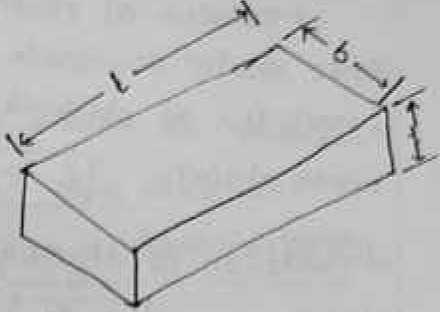
Volumetric Strain:-

When a member is subjected to force it undergoes change in dimensions hence the volume of member changes. The ratio of change in volume to original volume is called as volumetric strain.

$$\text{volumetric strain } (e_v) = \frac{\text{change in volume}}{\text{original volume}}$$

Volumetric strain of a Rectangular Bar:-

Let a rectangular bar having length = l, breadth = b & thickness = t. They undergo small change of Δl, Δb & Δt.



Original volume : $l \times b \times t$

Final volume = $(l+\Delta l) \times (b+\Delta b) \times (t+\Delta t)$

$= l\times b\times t + \Delta l\times b\times t + l\times t\times \Delta b + \cancel{\Delta l\times \Delta b\times t} + l\times b\times \Delta t + \cancel{b\times \Delta t\times \Delta l} + \cancel{l\times \Delta b\times \Delta t} + \cancel{\Delta l\times \Delta b\times \Delta t}$ [cancelled as very small]

$= lbt + b\times t\times \Delta l + l\times t\times \Delta b + lb\times \Delta t$

change in volume = $b\times t\times \Delta l + l\times t\times \Delta b + l\times b\times \Delta t$

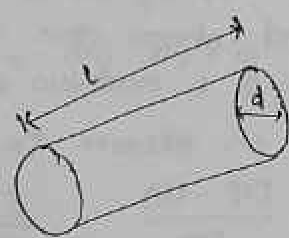
volumetric strain = $\dfrac{b\times t\times \Delta l + l\times t\times \Delta b + l\times b\times \Delta t}{lbt}$

$= \dfrac{\Delta l}{l} + \dfrac{\Delta b}{b} + \dfrac{\Delta t}{t} = \boxed{e_l + e_b + e_t = e_v}$

volumetric strain = strain in length + strain in breadth + strain in thickness.

Volumetric strain of Circular bar:-

Consider a bar of length 'l' and diameter 'd'. and Δl and Δd are change in length and diameter.



Original volume = $\frac{\pi}{4} d^2 l$

Final volume = $\frac{\pi}{4}(d+\Delta d)^2 (l+\Delta l)$

$= \frac{\pi}{4}[d^2 + \cancel{\Delta d^2} + 2d\,\Delta d]\,[l+\Delta l]$

$= \frac{\pi}{4}[d^2 l + 2\cdot d\cdot \Delta d\cdot l + d^2 \Delta l + \cancel{2d\,\Delta d\,\Delta l}]$

change in volume = $\frac{\pi}{4}[d^2 l + 2\cdot d\cdot l\cdot \Delta d + d^2 \Delta l] - \frac{\pi}{4} d^2 l$

$= \frac{\pi}{4}[2\cdot d\cdot l\cdot \Delta d + d^2 \Delta l]$

volumetric strain = $\dfrac{\frac{\pi}{4}[2\cdot d\cdot l\cdot \Delta d + d^2 \Delta l]}{\frac{\pi}{4} d^2 l}$

$$\boxed{e_v = 2\cdot \frac{\Delta d}{d} + \frac{\Delta l}{l}}$$

volumetric strain = ~~change~~ strain in length + 2 × strain in diameter.

Elastic constants:-

Modulus of elasticity, modulus of rigidity and bulk modulus are three elastic constants.

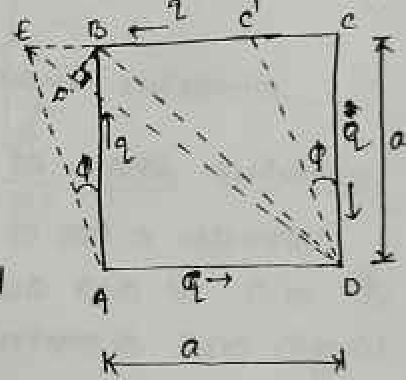
→ Modulus of elasticity:- It is the ratio between linear stress to linear strain. $\boxed{E = \sigma/e}$

→ Modulus of rigidity:- It is the ratio between shear stress to shear strain. $\boxed{G = q/\phi}$

→ Bulk modulus:- when a body subjected to identical stresses (σ) in all mutual perpendicular direction the body undergoes change in volume. So Bulk modulus is the ratio between identical stress in three mutually ⊥r direction to volumetric strain.

$$\boxed{K = \frac{\sigma}{e_v}}$$

Relationship between modulus of elasticity & modulus of rigidity:-



Consider a square element ABCD of sides 'a' subjected to pure shear of intensity 'q'. Due to application of load AEC'D is the deformed shape due to shear. Let the shear strain is ϕ & modulus of rigidity G.

Let us observe the strain along the diagonal BD.

$$= \frac{DE - DB}{DB} = \frac{EF}{DB} \quad \left[\begin{array}{l}\text{Drop a ⊥r BF on ED}\\ \text{so that BD = FD}\end{array}\right.$$

$$= \frac{EF}{AB\sqrt{2}} \quad [\text{As } DB = AB\sqrt{2}]$$

∵ Angle of deformation ϕ is very small so ∠BED can be assumed as 45°. So ∠BEF = 45°. Hence $EF = BE\cos 45°$

$$\text{strain in diagonal BD} = \frac{EF}{AB\sqrt{2}} = \frac{BE\cos 45°}{AB\sqrt{2}}$$

$$= \frac{a\tan\phi\,\cos 45°}{AB\sqrt{2}} = \frac{a\cdot\tan\phi\cdot 1}{a\cdot\sqrt{2}\cdot\sqrt{2}}$$

$$\left[\begin{array}{l}\triangle ABE\\ \tan\phi = \frac{BE}{AB}\\ \Rightarrow BE = AB\tan\phi\\ \quad = a\tan\phi\end{array}\right.$$

$$= \frac{1}{2}\tan\phi = \frac{1}{2}\phi \quad [\text{As } \phi \to 0, \tan\phi = \phi$$

$$= \frac{1}{2}\frac{q}{G} \quad \text{——— (1)}$$

Now Due to shear tensile stress is developed in diagonal BD and compressive stress is developed in AC. Let the tensile strain along the length BD is q

$$\text{Tensile strain along BD} = \frac{q}{E} + \mu\frac{q}{E} = \frac{q}{E}(1+\mu) \quad \text{——— (2)}$$

Equating eqⁿ ① & ②

$$\frac{1}{2} \times \frac{q}{G} = \frac{q}{E}(1+\mu)$$

$$\boxed{E = 2G(1+\mu)}$$

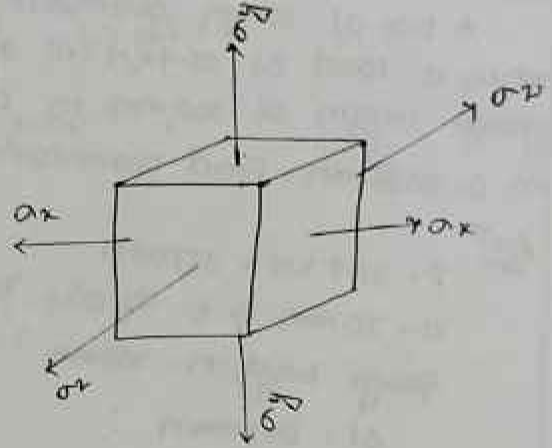
<u>Relationship between modulen of elasticity and Bulk modulen:</u>

Consider a cubic element subjected to stress σ in three mutually perpendicular direction x, y, z as shown.

Now the stress σ_x in x direction causes tensile $\frac{P}{E}$ in x-direction while stress σ_y & σ_z in y & z direction causes obstacle to σ_x.

So strain in x-direction

$$e_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}$$
$$= \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}$$
$$= \frac{\sigma}{E} - 2\mu\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\mu)$$



$\sigma_x = \sigma_y = \sigma_z = \sigma$

Similarly

$$e_y = \frac{\sigma}{E}(1-2\mu)$$
$$e_z = \frac{\sigma}{E}(1-2\mu)$$

volumetric strain

$$e_v = e_x + e_y + e_z$$
$$= \frac{3\sigma}{E}(1-2\mu) \quad \text{——— ①}$$

$$\text{Bulk modulen}(K) = \frac{\sigma}{e_v} = \frac{\sigma}{\frac{3\sigma}{E}(1-2\mu)} = \frac{E}{3(1-2\mu)}$$

$$\Rightarrow \boxed{E = 3K(1-2\mu)} \quad \text{——— ②}$$

Now

$$E = 2G(1+\mu), \quad E = 3K(1-2\mu)$$

$$\mu = \frac{E}{2G} - 1$$

Now, $E = 3K(1-2\mu)$

$$E = 3K\left[1 - 2\left(\frac{E}{2G} - 1\right)\right] = 3K\left[1 - \frac{2E}{2G} + 2\right]$$

$$\boxed{\cancel{E = 3K\left[\frac{-E}{G} + 3\right] = 9K + \frac{3KE}{G}}}$$

$$E = 3K\left[3 - \frac{E}{G}\right] = 9K - \frac{3KE}{G}$$

$$\Rightarrow 9K = E + \frac{3KE}{G} = E\left(1 + \frac{3K}{G}\right) = E\left(\frac{G+3K}{G}\right)$$

$$\Rightarrow E = \boxed{\frac{9KG}{G+3K}} \Rightarrow \frac{9}{E} = \frac{G+3K}{KG}$$

$$\Rightarrow \boxed{\frac{9}{E} = \frac{1}{K} + \frac{3}{G}}$$

Problem:-

A bar of 20mm diameter is tested in tension. It is observed that when a load of 37.7KN is applied, the extension measured over a gauge length of 200mm is 0.12mm and contraction in diameter is 0.0036 mm. Find poisson's ratio and elastic constants.

Sol^n

$P = 37.7\ KN = 37700\ N$

$d = 20\ mm \Rightarrow A = \frac{\pi}{4}d^2 = \frac{\pi}{4}\times(20)^2 = 100\pi\ mm^2$

Gauge length (l) = 200mm

$\Delta l = 0.12mm$

$\Delta d = 0.0036mm$

$$\text{linear strain} = \frac{\Delta l}{l} = \frac{0.12}{200} = 0.0006$$

$$\text{Lateral strain} = \frac{\Delta d}{d} = \frac{0.0036}{20} = 0.00018$$

$$\text{poisson's ratio } (\mu) = \frac{\text{Lateral strain}}{\text{Linear strain}} = \frac{0.0006}{0.00018} = \boxed{0.3}$$

we know,

$$\Delta l = \frac{PL}{AE} \Rightarrow E = \frac{PL}{\Delta L \cdot A} = \frac{37700 \times 200}{0.12 \times 100\pi} = \boxed{200004.71\ N/mm^2}$$

Now $E = 2G(1+\mu) \Rightarrow G = \frac{E}{2(1+\mu)} = \frac{200004.71}{2(1+0.3)} = \boxed{76924.89\ N/mm^2}$

we get $G = \frac{E}{2(1+\mu)} = \frac{200004.71}{2(1+0.3)} = \boxed{76924.89\ N/mm^2}$

From the relation

$$E = 3K(1-2\mu)$$

$$K = \frac{E}{3(1-2\mu)} = \frac{200004.71}{3(1-2\times0.3)} = \boxed{166670.59\ N/mm^2}$$

problem:-

A circular rod of 100 mm diameter and 500 mm long is subjected to a tensile force of 1000 kN. Determine modulus of rigidity, bulk modulus and change in volume if poisson's ratio = 0.3 and young's modulus E = 2×10^5 N/mm².

Soln $E = 2G(1+\mu) \Rightarrow G = \frac{E}{2(1+\mu)} = \frac{2\times10^5}{2(1+0.3)} = 0.7692\times10^5\ N/mm^2$

$E = 3K(1-2\mu) \Rightarrow K = \frac{E}{3(1-2\mu)} = \frac{2\times10^5}{3\times(1-2\times0.3)} = 1.667\times10^5\ N/mm^2$

Longitudinal stress = $P/A = \frac{1000\times10^3}{\pi/4\times100^2} = 127.324\ N/mm^2$

Linear strain $(e_x) = \frac{\text{stress}}{E} = \frac{127.324}{2\times10^5} = 63.662\times10^{-5}$

Lateral strain $(e_y) = -\mu e_x$, $e_z = -\mu e_x$

volumetric strain $(e_v) = e_x + e_y + e_z$
$= e_x(1-2\mu)$
$= 63.662\times10^{-5}\,(1-2\times0.3)$
$= 25.4648\times10^{-5}$

But $e_v = \frac{\text{change in volume}}{\text{Original volume}}$

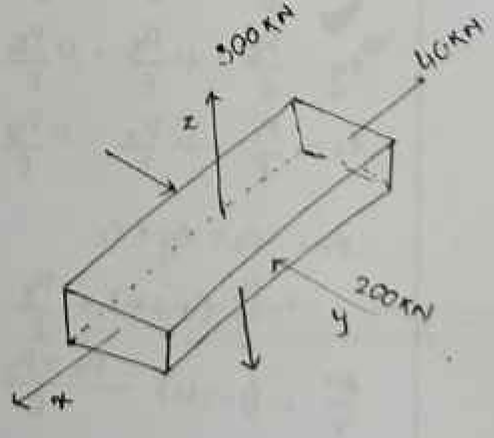
change in volume $= e_v\times V = 25.4648\times10^{-5}\times\frac{\pi}{4}\times d^2\times l$
$= 1000\ mm^2$

problem:-

A 500 mm long rectangular bar of cross section 20mm × 40mm. This bar is subjected to

(i) 40 kN Tensile force on 20mm × 40mm faces.
(ii) 200 kN compressive force on 40mm × 500mm faces.
(iii) 300 kN tensile force on 40mm × 500 mm faces.
Find the change in volume if $E = 2\times10^5$ N/mm² and $\mu = 0.3$

Soln All the forces in member are mutually perpendicular.



So $\sigma_x = \frac{P_x}{A_x} = \frac{40\times10^3}{20\times40} = 50\ N/mm^2$

$\sigma_y = \frac{P_y}{A_y} = \frac{200\times10^3}{20\times500} = 20\ N/mm^2$

$\sigma_z = \frac{P_z}{A_z} = \frac{300\times10^3}{40\times500} = 15\ N/mm^2$

strain along 'x' direction

$$e_x = \frac{\sigma_x}{E} = \frac{\sigma_x}{E} + \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$

$$= \frac{1}{E}\left[50 + 0.3\times 20 - 0.3\times 15\right] = \frac{51.5}{E}$$

$$e_y = \frac{\sigma_y}{E} = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E}$$

$$= \frac{1}{E}\left[20 - \mu\times 50 - 0.3\times 15\right] = \frac{-39.5}{E}$$

$$e_z = \frac{\sigma_z}{E} = \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} + \mu\frac{\sigma_y}{E}$$

$$= \frac{1}{E}\left[15 - \mu\times 50 + 0.3\times 20\right] = \frac{6}{E}$$

volumetric strain $e_v = e_x + e_y + e_z$

$$= \frac{1}{E}\left[51.5 - 39.5 + 6\right] = \frac{18}{E}$$

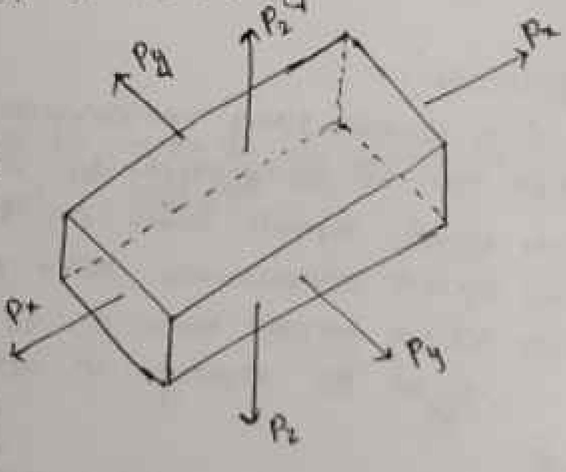
But $\frac{\text{change in volume}}{\text{Original volume}} = e_v$

change in volume $= e_v \times$ volume $= \frac{18}{2\times10^5}\times 20\times 40\times 500$

$= \boxed{36\,mm^2}$

problem:-

A bar of rectangular section shown in fig is subjected to stress P_x, P_y and P_z in x, y and z directions respectively. Show that if the sum of these stresses is zero, there is no change in volume in the bar.



Soln

All the forces are acting mutually perpendicular to each other.

Hence,

$$e_x = \frac{P_x}{E} - \mu\frac{P_y}{E} - \mu\frac{P_z}{E} = \boxed{\frac{P_x}{E} / -\frac{P_y}{E} / -\frac{P_z}{E}}$$

$$e_y = \frac{P_y}{E} - \mu\frac{P_x}{E} - \mu\frac{P_z}{E} = \boxed{\frac{P_y}{E} / -\frac{P_x}{E} / -\frac{P_z}{E}}$$

$$e_z = \frac{P_z}{E} - \mu\frac{P_x}{E} - \mu\frac{P_y}{E} = \boxed{\frac{P_z}{E} / -\frac{P_x}{E} / -\frac{P_y}{E}}$$

$$e_v = e_x + e_y + e_z$$

$$= \frac{P_x}{E}(1-2\mu) + \frac{P_y}{E}(1-2\mu) + \frac{P_z}{E}(1-2\mu)$$

$$\frac{\Delta v}{v} = (1-2\mu)\frac{P_x + P_y + P_z}{E}$$

If $P_x + P_y + P_z = 0$

$e_v = \frac{\Delta V}{V} = 0$

$\Rightarrow \boxed{\Delta V = 0}$

i.e. there is no volumetric change if $P_x + P_y + P_z = 0$

problem:-

In a tensile test young's modulus of mat. is found to be 2.1×10^5 N/mm². On the same material torsion test is conducted and modulus of rigidity is found to be 0.78×10^5 N/mm². Determine poisson's ratio and bulk modulus of the material.

Soln $E = 2.1 \times 10^5$ N/mm², $G = 0.78 \times 10^5$ N/mm²

Now, $E = 2G(1+\mu)$

$\Rightarrow 2.1 \times 10^5 = 2 \times 0.78 \times 10^5 (1 + \mu)$

$\Rightarrow (1+\mu) = 1.346 \Rightarrow \boxed{\mu = 0.346}$

Again $E = 3K(1-2\mu) \Rightarrow 2.1 \times 10^5 = 3K(1 - 2 \times 0.346)$

$\Rightarrow K = \frac{2.1 \times 10^5}{3 \times 0.308} \Rightarrow \boxed{K = 2.275 \times 10^5 \text{ N/mm}^2}$

problem:-

A material has modulus of rigidity equal to 0.4×10^5 N/mm² and bulk modulus equal to 0.75×10^5 N/mm². Find its young's modulus and poisson's ratio.

Soln
$G = 0.4 \times 10^5$ N/mm², $K = 0.75 \times 10^5$ N/mm²

Using the relation -

$E = \frac{9KG}{3K+G} \Rightarrow \frac{9 \times 0.75 \times 10^5 \times 0.4 \times 10^5}{3 \times 0.75 \times 10^5 + 0.4 \times 10^5} = \frac{2.7 \times 10^{10}}{2.65 \times 10^5} = \boxed{1.01 \times 10^5 \text{ N/mm}^2}$

Again,

$E = 2G(1+\mu)$

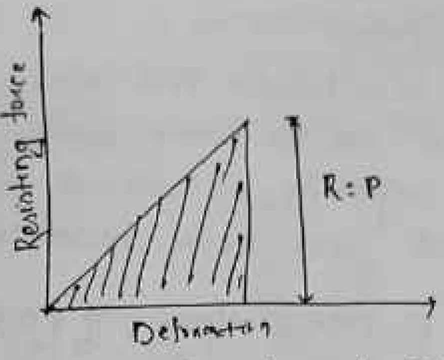
$\Rightarrow 1.01 \times 10^5 = 2 \times 0.4 \times 10^5 (1+\mu)$

$\Rightarrow (1+\mu) = 1.26$

$\Rightarrow \boxed{\mu = 0.26}$

~~St. energy:-~~ Strain Energy:-

When the external force is applied on a body the body offers resistance. and it developes fully when deformation is fully developed. As the resistance force increases with deformation the work is done by the force. This force is stored in the body in terms of energy. when the load is removed this energy brings the body back.



So the energy stored in the body during straining is called strain energy.

Consider a member of length 'l' and cross sectional area 'A' be is subjected to a force 'P'. Let the resistance at any moment be 'R'.

when deformation is zero, $R = 0$.

when deformation is $\Delta l = el$, $R = P$

work done by resisting force = Avg. resistance $\times \Delta l$

$$= \frac{P+0}{2} \times el = \frac{1}{2} P e l$$

$$= \frac{1}{2} \sigma A e l = \boxed{\frac{1}{2} \sigma e V} \quad \text{——} \ (1)$$

Strain energy in the body = W.D by resisting force

$$S.E = \frac{1}{2} \sigma e V = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

$$= \frac{1}{2} \sigma \cdot \frac{\sigma}{E} V = \boxed{\frac{1}{2} \frac{\sigma^2 V}{E}} \quad \text{——} \ (2)$$

Resilience:-

Strain energy per unit volume is called Resilience

$$\boxed{\text{Resilience} = \frac{\sigma^2}{2E}} \quad \text{——} \ (3)$$

Proof Resilience:-

The maximum strain energy which can be stored by a body without undergoing permanent deformation is called proof Resilience.

$$\boxed{\text{proof Resilience} = \frac{\sigma_y^2}{2E}} \quad \text{——} \ (4)$$

[Max^m stress = yield stress (σ_y).

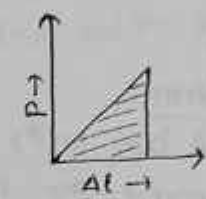
(ii)

Stress analysis due to various types of loads can be done by strain energy method. In this method strain energy is equated to work done by the loads.

(i) Gradually applied load:-

Consider a bar of length L, cross sectional area A subjected to load P. If the load P is gradually applied, the load increases from 0 to P as extension increases from 0 to Δl gradually. Hence workdone by load

$= P_{avg} \times \Delta L = \frac{0+P}{2} \times \Delta l = \frac{P}{2} e L$

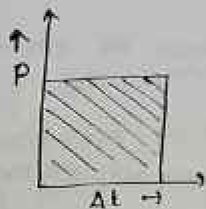


Equating with eq^n ①

$\frac{1}{2} \sigma e V = \frac{P}{2} e L \Rightarrow \sigma \cancel{e} A \cancel{L} = P \cancel{e} \cancel{L} \Rightarrow \boxed{\sigma = P/A}$

(ii) Suddenly applied load:-

Due to the suddenly applied load the stress and strain changes instantly. Suddenly applied load P means P is acting when $\Delta l = 0$ and acts throughout the period of extension.



Hence w.D by load $= P \times \Delta l = P \times e \times L$

Equating with eq^n ①

$\frac{1}{2} \sigma e V = P \times e \times L \Rightarrow \frac{1}{2} \sigma \cancel{e} \times A \times \cancel{L} = P \times \cancel{e} \times \cancel{L}$

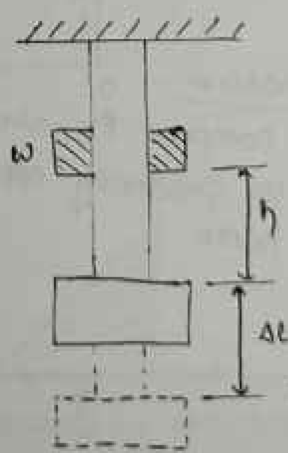
$\Rightarrow \boxed{\sigma = \frac{2P}{A}}$

Note:-

Stress developed due to suddenly applied load is twice than the gradually applied load.

(iii) Impact load / Freely falling load:-

In this type of loading, the load falls through a ~~load~~ height 'h' before it acts on the body. Fig shown such a loading case on a bar of length L, in which load W falls through a height h. After striking the collar the load travels a distance Δl



work done by load $= W(h + \Delta l)$

$= W[h + eL] = W[h + \frac{\sigma}{E} L]$

Strain energy stored $= \frac{1}{2} \frac{\sigma^2}{E} AL$

Equating both

$W[h + \frac{\sigma}{E} L] = \frac{1}{2} \frac{\sigma^2}{E} AL$

$\sigma^2 = \frac{2EW}{AL}[h + \frac{\sigma}{E} L] = \frac{2EWh}{AL} + \frac{2EW\sigma L}{AEL}$

$$\sigma^2 - \frac{2W}{A}\sigma - \frac{2EWh}{AL} = 0$$

Equating the equation in the form of $ax^2 + bx + c = 0$

$$\sigma_{1,2} = \frac{1}{2}\left[-\left(-\frac{2W}{A}\right) \pm \sqrt{\left(\frac{2W}{A}\right)^2 + 4\frac{2EWh}{AL}}\right] \qquad \left[x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right.$$

$$\Rightarrow \left[\sigma = \frac{1}{2}\left[\frac{2W}{A} + \sqrt{\left(\frac{2W}{A}\right)^2 + 4\frac{2EWh}{AL}}\right]\right]$$

Negative solution is not considered as we deal with the maximum stress value.

$$\Rightarrow \sigma = \frac{1}{2}\left[\frac{2W}{A} + \frac{2W}{A}\sqrt{1 + 4\frac{2EWh}{AL}\frac{A^2}{4W^2}}\right]$$

$$= \frac{1}{2}\times\frac{2W}{A}\left[1 + \sqrt{1 + \frac{2EhA}{LW}}\right]$$

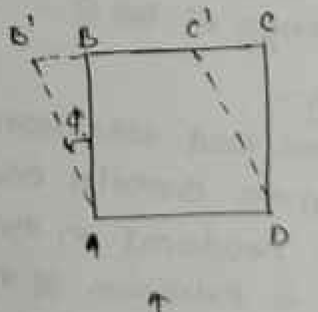
$$\boxed{\sigma = \frac{W}{A}\left[1 + \sqrt{1 + \frac{2EhA}{LW}}\right]}$$

(ii) Shock load:-

Shock load is measured in terms of energy. So shock load applied in a member let be U unit. The stress produced in the bar can be obtained by equating the strain energy with the energy produced by shock load.

$$\frac{\sigma^2}{2E}AL = U \Rightarrow \boxed{\sigma = \sqrt{\frac{2EU}{AL}}}$$

(iv) Strain energy due to shear stress:-

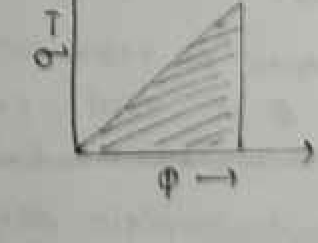


Consider an element ABCD of thickness 't' is subjected to shearing stress 'q'. The internal strain developed is zero when $q = 0$ and is ϕ when $q = q$.

Shearing force Q acting on face BC:

$$Q = q \times BC \times t$$

workdone by the force = $Q_{avg} \times BB'$



$$= \frac{0 + Q}{2} \times AB \times \phi \qquad \left[BB' = AB\tan\phi \right.$$
$$\qquad\qquad\qquad\qquad = AB \times \phi$$

$$= \frac{Q}{2} \times AB \times \phi$$

$$= \frac{1}{2} \times q \times BC \times t \times AB \times \phi$$

$$= \frac{1}{2} \times q \times \phi \times (BC \times AB \times t)$$

work done = $\frac{1}{2}$ × shear stress × shear strain × volume.

strain energy stored = work done by internal stresses

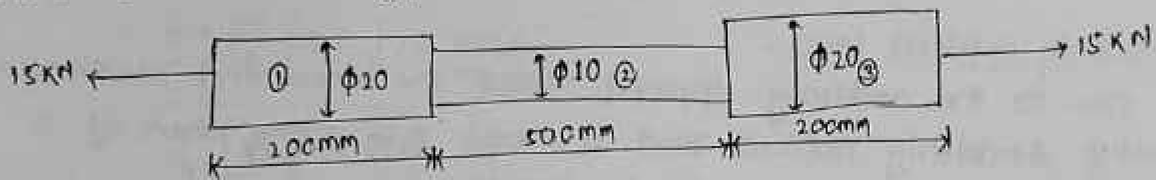
$$= \frac{1}{2} q \times \phi \times V$$

$$= \frac{1}{2} q \times \frac{q}{G} \times V \quad \left[\phi = \frac{q}{G} \right]$$

$$= \boxed{\frac{1}{2} \frac{q^2 V}{G}}$$

problem:-

A bar with circular cross section is subjected to a load of 15KN. Determine the strain energy stored in it take $E = 2 \times 10^5$ N/mm²



Sol

Stress in cross section ① & ③ $= \sigma_1 = \sigma_3 = P/A_1 = \frac{15 \times 1000}{\frac{\pi}{4} \times (20)^2} = 47.77$ N/mm²

Stress in cross section ② $= P/A_2 = \frac{15 \times 1000}{\frac{\pi}{4} \times (10)^2} = 191.08$ N/mm²

strain energy stored in the member = Energy at ① + Energy at ② + Energy at ③

$$= \frac{(\sigma_1)^2 V_1}{2E} + \frac{(\sigma_2)^2 V_2}{2E} + \frac{(\sigma_3)^2 V_3}{2E}$$

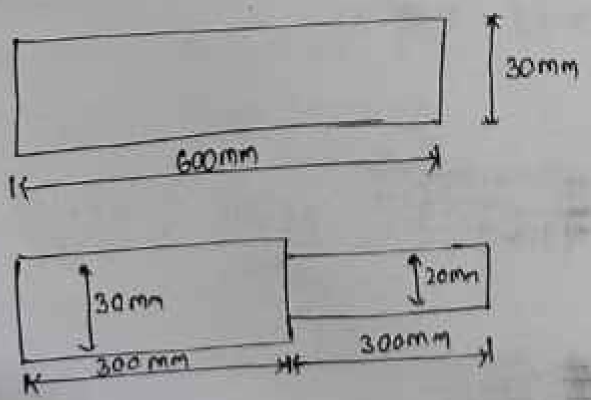
$$= \frac{1}{2E} \left[2 \times \sigma_1^2 V_1 + \sigma_2^2 V_2 \right] \quad \left[\text{As } \sigma_1 = \sigma_3, \ V_1 = V_3 \right]$$

$$= \frac{1}{2 \times 2 \times 10^5} \left[2 \times (47.77)^2 \times \frac{\pi}{4} \times (20)^2 \times 200 + (191.08)^2 + \frac{\pi}{4} \times (10)^2 \times 500 \right]$$

$$= \boxed{4299.23 \text{ N-mm}}$$

problem:-

Compare the strain energy of two bars of same material as shown for (a) Gradually applied load is same (b) Maximum stress produced is same



Sol^n

(i) when applied load is same:-

Strain energy stored in bar ① $= \frac{\sigma^2}{2E} \times vol = \frac{1}{2E}\left[\frac{P}{\pi/4 (30)^2}\right]^2 \times \pi/4 \times (30)^2 \times 600$

$= \frac{P^2}{E} \times 0.4244$

Strain energy stored in bar ② = Energy at ① + Energy at ②

$\Rightarrow \frac{\sigma_1^2}{2E} v_1 + \frac{\sigma_2^2}{2E} v_2 \Rightarrow \frac{1}{2E}\left[\left(\frac{P}{\pi/4 (30)^2}\right)^2 \times \frac{\pi}{4}(30)^2 \times 300 + \left(\frac{P}{\pi/4 (20)^2}\right)^2 + \frac{\pi}{4} \times (20)^2 \times 200\right]$

$= \frac{P^2}{E}[0.2122 + 0.4775] = 0.6897 \frac{P^2}{E}$

$\frac{\text{Energy at bar ①}}{\text{Energy at bar ②}} = \frac{0.4244 \times P^2/E}{0.6897 \times P^2/E} = \boxed{0.615}$

(ii) when maximum stress produced is same:-

Let maximum stress is σ, so maximum stress at bar ① $= \sigma$

Now, strain energy at bar ① $= \frac{\sigma^2}{2E} \times \frac{\pi}{4} \times (30)^2 \times 600 = \frac{67500\sigma^2\pi}{E}$

Bar-2

Maximum stress is at diameter 20mm i.e. $\sigma_2 = \sigma$

Stress at dia 30 mm $= \sigma_1 = \frac{\sigma \times A_2}{A_1} = \frac{\sigma \times \pi/4 \times (20)^2}{\pi/4 \times (30)^2} = 4/9\,\sigma$

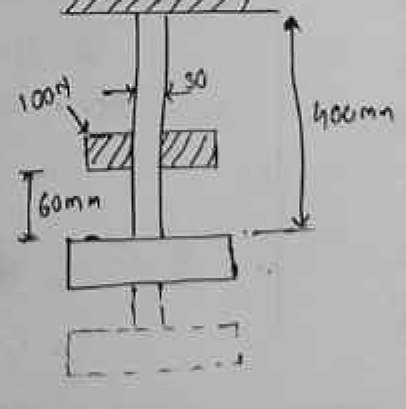
Strain energy at bar ② $= \frac{\sigma_1^2 v_1}{2E} + \frac{\sigma_2^2 v_2}{2E}$

$= \frac{(4/9\,\sigma)^2}{2E} \times \frac{\pi}{4} \times (30)^2 \times 300 + \frac{\sigma^2}{2E} \times \frac{\pi}{4} \times (20)^2 \times 300 = \frac{\pi\sigma^2}{E}\, 21666.67$

$\frac{\text{Energy at bar ①}}{\text{Energy at bar ②}} = \frac{67500}{21666.67} = \boxed{3.11}$

problem:-

A 100N load falls from a height of 60mm on a collar attached to a bar of 30 mm diameter and 400mm long. Find the instantaneous stress and extension produced in the bar. Take $E = 2 \times 10^5 N/mm^2$. What is the %age of error if extension of the bar is neglected in final work done by the load?



Sol^n Area of cross section (A) $= \frac{\pi}{4} \times 30^2 = 225\pi\, mm^2$

$W = 100N$, $E = 2 \times 10^5 N/mm^2$

$h = 60mm$, $L = 400mm$.

→ Instantaneous stress produced due to impact load $= (\sigma) = \frac{W}{A}\left[1 + \sqrt{1 + \frac{2AEh}{WL}}\right]$

$$= \frac{100}{225\pi}\left[1+\sqrt{1+\frac{2\times 225\pi\times 2\times10^5\times 60}{100\times 400}}\right] = \boxed{92.273\,N/mm^2}$$

→ If the extension of bar is neglected

$$\frac{\sigma^2}{2E}AL = w\times h \Rightarrow \sigma = \sqrt{\frac{w\times h\times 2\times E}{A\times L}}$$

$$= \sqrt{\frac{100\times 60\times 2\times 2\times 10^5}{225\pi\times 400}} = \boxed{92.132\,N/mm^2}$$

→ %age error in approximating

$$= \frac{92.273-92.132}{92.273}\times 100 = \boxed{0.153\%}$$

→ Instantaneous extension produced

$$= \frac{\sigma\times L}{E} = \frac{92.132\times 400}{2\times 10^5} = \boxed{0.1842\,mm}$$

⑤ :- THIN CYLINDRICAL AND SPHERICAL SHELL :-

Introduction:-

Cylinders and spheres are commonly used as components of machineries and chemical plants and usually they are subjected to internal and external pressures. Cylinders can be classified in to two catagories

→ Thin cylinders or spherical shell :- when the thickness is less than 1/10th to 1/15th of its radius or 1/20th of diameter then it is called thin cylinder. Here radial stress is small so neglected.

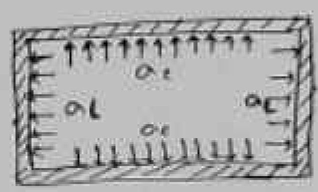
→ thick cylinders and spherical shell:- when the thickness is more than 1/10th of radius or 1/20th of diameter then it is called thick cylinder. Radial stress is taken in to consideration.

Stresses in thin cylindrical shells:-

A thin cylindrical shell is subjected to following stresses.

(i) Circumferencial stress or hoop stress (σ_c)

(ii) Longitudinal stress (σ_L)



Note:-
As the thickness is very small so radial stress is neglected.

(i) Longitudinal ~~circumferencial~~ stress:-

Force acting on the walls of the cylinder

$$F_P = P \times \frac{\pi}{4} d^2 = p \times Area \quad \text{——} ①$$

Resisting force at the cylinder wall

$$F_R = \pi d t \times \sigma_L \quad \text{——} ②$$

For equilibrium

$$F_P = F_R$$

$$\Rightarrow p \times \frac{\pi}{4} d^2 = \pi d \times t \times \sigma_L$$

$$\Rightarrow \boxed{\sigma_L = \frac{Pd}{4t}} \quad \text{——} ③$$

P = Inside pressure of fluid
d = Dia of cylinder.
t = thickness of cylinder
σ_L = stress of length ~~circumference~~

The stress which is acting along the length of the cylinder is called longitudinal stress.

(i)

ii) Circumferential stress (σ_c):

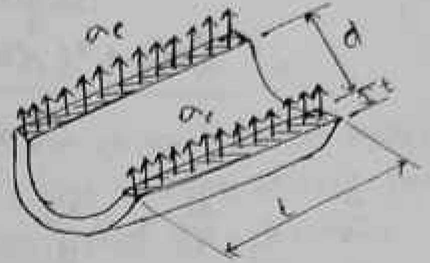
The stress which is acting along the circumference is called circumferential stress.

Fluid force acting due to pressure:

$$F_p = d \times l \times P \quad \text{———} \ (4)$$

Resisting force acting on cylinder walls:

$$F_r = 2 \times t \times l \times \sigma_c \quad \text{———} \ (5)$$



For equilibrium, $F_p = F_r$

$$\Rightarrow d \times l \times P = 2 \times t \times l \times \sigma_c$$

$$\Rightarrow \boxed{\sigma_c = \frac{P \times d}{2t}} \quad \text{———} \ (6)$$

P = Pr of fluid

t = thickness

l = Length of cylinder

Note:-

while designing the cylindrical shell circumferential stress is considered because its value is more than longitudinal stress.

Change in dimension of cylindrical shell:-

we know that $\sigma_c = \frac{Pd}{2t}$ & $\sigma_l = \frac{Pd}{4t}$

Now, circumferential strain (e_c) $= \frac{\sigma_c}{E} - \mu \frac{\sigma_l}{E}$

$$= \frac{1}{E}\left[\frac{Pd}{2t} - \mu \frac{Pd}{4t}\right] = \frac{Pd}{4tE}\left[2 - \mu\right] \quad \text{———} \ (7)$$

Let 'δd' be the change in dimension

$$\text{So } e_c = \left[\frac{\text{Final circumference} - \text{Original circumference}}{\text{Original circumference}}\right]$$

$$= \left[\frac{\pi(d + \Delta d) - \pi d}{\pi d}\right] = \frac{\Delta d}{d} \quad \text{———} \ (8)$$

Equating (7) & (8)

$$\frac{\Delta d}{d} = \frac{Pd}{4tE}\left[2 - \mu\right] \quad \text{———} \ (9)$$

Again, longitudinal strain (e_l) $= \frac{\sigma_l}{E} - \mu \frac{\sigma_c}{E}$

$$= \frac{1}{E}\left[\frac{Pd}{4t} - \mu \frac{Pd}{2t}\right] = \frac{Pd}{4tE}\left[1 - 2\mu\right] \quad \text{———} \ (10)$$

Let 'Δl' be the change in length

$$\text{So } e_l = \frac{\Delta l}{l} = \frac{Pd}{4tE}\left[1 - 2\mu\right] \quad \text{———} \ (11)$$

change in volume = Final volume − Original volume

$$= \frac{\pi}{4}(d + \Delta d)^2 (l + \Delta l) - \frac{\pi}{4} d^2 l$$

$$= \frac{\pi}{4}\left[d^2 + \Delta d^2 + 2 \times d \times \Delta d\right]\left[l + \Delta l\right] - \frac{\pi}{4} d^2 l$$

$= \frac{\pi}{4}\left[d^2 l + d^2 \Delta l + 2 \times d \times \Delta d \times l + \cancel{2 \times d \times \Delta d \times \Delta l}\right] - \frac{\pi}{4} d^2 l$

volumetric strain = $\frac{\text{change in volume}}{\text{Original volume}}$

$$= \frac{\frac{\pi}{4} d^2 l + \frac{\pi}{4} d^2 \Delta l + \frac{\pi}{4} 2 \times d \times \Delta d \times l - \frac{\pi}{4} d^2 l}{\frac{\pi}{4} d^2 l}$$

$$\frac{\Delta v}{v} = \frac{\Delta l}{l} + 2\frac{\Delta d}{d} = \boxed{e_L + 2e_c} \quad \text{——— (12)}$$

putting the values of eqn (9) & (11)

$$\frac{\Delta v}{v} = \frac{Pd}{4tE}(1-2\mu) + 2\frac{Pd}{4tE}(2-\mu)$$

$$= \frac{Pd}{4tE}[1-2\mu+4-2\mu] = \boxed{\frac{Pd}{4tE}[5-4\mu]} \quad \text{——— (13)}$$

problem:-

A cylindrical shell 2.5 m long which is closed at its ends has an internal diameter of 1 m and a wall thickness of 12mm. calculate the circumferencial and longitudinal stresses induced and also the change in dimensions of the shell if it is subjected to an internal pressure of 1.8MN/m². Take E = 200GN/m² and poisson's ratio(μ) = 0.25

sol

given data

$L = 2.5m = 2500mm$, $d = 1m = 1000mm$, $t = 12mm$, $P = 1.8MN/m^2 = 1.8 N/mm^2$

$E = 200GN/m^2 = 2\times10^5 N/mm^2$, $\mu = 0.25$

→ circumferencial stress (σ_c) = $\frac{Pd}{2t} = \frac{1.8\times1000}{2\times12} = \boxed{75 N/mm^2}$

→ Longitudinal stress (σ_L) = $\frac{Pd}{4t} = \frac{1.8\times1000}{4\times12} = \boxed{37.5 N/mm^2}$

→ Diametrical strain (e_c) = $\frac{\Delta d}{d} = \frac{Pd}{4tE}(2-\mu) = \frac{1.8\times1000}{4\times12\times2\times10^5}(2-0.25)$

$= \boxed{3.28\times10^{-4}}$

→ Longitudinal strain (e_L) : $\frac{\Delta l}{l} = \frac{Pd}{4tE}(1-2\mu) = \frac{1.8\times1000}{4\times12\times2\times10^5}(1-2\times0.25)$

$= \boxed{9.375\times10^{-5}}$

→ change in diameter(Δd) = $e_c \times d = 3.28\times10^{-4}\times 1000 = \boxed{0.328mm}$

→ change in length (Δl) = $e_L \times L = 9.375\times10^{-5}\times2500 = \boxed{0.234mm}$

→ volumetric strain (e_v) = $e_L + 2e_c = 9.375\times10^{-5} + 2\times3.28\times10^{-4}$

$= \boxed{7.497\times10^{-4}}$

→ change in volume $(\Delta V) = e_v \times V = 7.497 \times 10^{-4} \times \frac{\pi}{4} \times d^2 \times L$

$= 7.497 \times 10^{-4} \times \frac{\pi}{4} \times (1000)^2 \times 2000$

$= \boxed{1177704.54\ mm^3}$

Problem:-

A thin cylinderical shell, 2m long has 200mm diameter and thickness of metal 10mm. It is filled completely with a fluid at atmospheric pressure If an additional 25000 mm³ fluid is pumped in, find the pressure developed and hoop & longitudinal strain developed. Find also change in diameter and length. Take $E = 2\times10^5 N/mm^2$, $\mu = 0.3$

Soln

Given

$L = 2m = 2000mm$, $d = 200mm$, $t = 10mm$, $\Delta V = 25000\ mm^3$, $E = 2\times10^5 N/mm^2$, $\mu = 0.3$

So $\sigma_c = \frac{pd}{2t} = \frac{p\times200}{2\times10} = 10P$

$\sigma_L = \frac{pd}{4t} = \frac{p\times200}{4\times10} = 5P$

Circumferencial strain $(e_c) = \frac{pd}{4tE}(2-\mu) = \frac{p\times200}{4\times10\times2\times10^5}(2-0.3) = 4.25\times10^{-5}P$

Longitudinal strain $(e_L) = \frac{pd}{4tE}(1-2\mu) = \frac{p\times200}{4\times10\times2\times10^5}(1-2\times0.3) = 1\times10^{-5}P$

volumetric strain $(e_v) = \frac{\Delta V}{V} = 2e_c + e_L = 2\times4.25\times10^{-5}p + 1\times10^{-5}p$

$= 9.5\times10^{-5}p$

$\cancel{p = \frac{V\times9.5\times10^{-5}}{\Delta V}}$

$p = \frac{\Delta V}{V\times9.5\times10^{-5}} = \frac{25000}{62831853.07\times9.5\times10^{-5}}$

$V = \frac{\pi}{4}\times d^2\times L$
$= \pi/4\times(200)^2\times2000$
$= 62831853.07$

$= \boxed{4.18\ N/mm^2}$

$\sigma_c = 10P = 10\times4.18 = \boxed{41.8\ N/mm^2}$

$\sigma_L = 5P = 5\times4.18 = \boxed{20.9\ N/mm^2}$

change in diameter $(\Delta d) = e_c\times d = 4.25\times10^{-5}\times P\times d = 4.25\times10^{-5}\times4.18\times200$

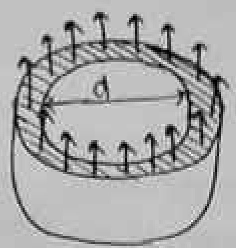
$= \boxed{0.035\ mm}$

change in length $(\Delta L) = e_L\times L = 1\times10^{-5}\times P\times L = 1\times10^{-5}\times4.18\times2000$

$= \boxed{0.084\ mm}$

Thin spherical shell:-

Force exerted by internal fluid at the junction $(F_p) = \frac{\pi}{4} d^2 \times P$ —— (14)



Resistance force exerted by the shell

$(F_r) = \pi \times d \times t \times \sigma_t$ —— (15)

For equilibrium eq^n (14) = eq^n (15)

$\pi \times d \times t \times \sigma_t = \frac{\pi}{4} \times d^2 \times P$

$$\boxed{\sigma_t = \frac{Pd}{4t}}$$ —— (16)

Strain:-

circumferencial strain $(e_c) = \frac{\Delta d}{d} = \frac{\sigma_t}{E} - \mu\frac{\sigma_t}{E} = \frac{\sigma_t}{E}(1-\mu) = \boxed{\frac{Pd}{4tE}(1-\mu)}$

volumetric strain $(e_v) = \frac{\Delta V}{V} = \frac{\text{Final volume} - \text{Original volume}}{\text{Original volume}}$

$$= \frac{\frac{\pi(d + \Delta d)^3}{6} - \frac{\pi d^3}{6}}{\frac{\pi d^3}{6}} = \frac{\cancel{d^3} + \cancel{3 \cdot d(\Delta d)^2} + 3d^2\Delta d + \cancel{\Delta d^3} - \cancel{d^3}}{d^3}$$

$$= \frac{3d^2 \Delta d}{d^3} = \boxed{3\frac{\Delta d}{d}} = 3e_c$$

So $\boxed{e_v = \frac{3Pd}{4tE}(1-\mu)}$

problem:-

At atmospheric pressure, a thin spherical shell has diameter 750 mm and thickness 10 mm. Find the strain induced and change in diameter and volume when fluid pressure is increased to 2.5 N/mm^2. Take $E = 2\times10^5$ N/mm^2, $\mu = 0.35$

Sol: $D = 750$ mm, $t = 10$ mm, $P = 2.5$ N/mm^2, $E = 2\times10^5$ N/mm^2, $\mu = 0.35$

→ Hoop stress $(\sigma_t) = \frac{PD}{4t} = \frac{750 \times 2.5}{4\times 10} = \boxed{46.875 \text{ N/mm}^2}$

→ circumferencial strain $(e_c) = \frac{\Delta d}{d} = \frac{Pd}{4tE}(1-\mu)$

change in Diameter ⇒ $\Delta d = \frac{Pd^2}{4tE}(1-\mu) = \frac{2.5\times(750)^2}{4\times10\times2\times10^5}(1-0.35)$

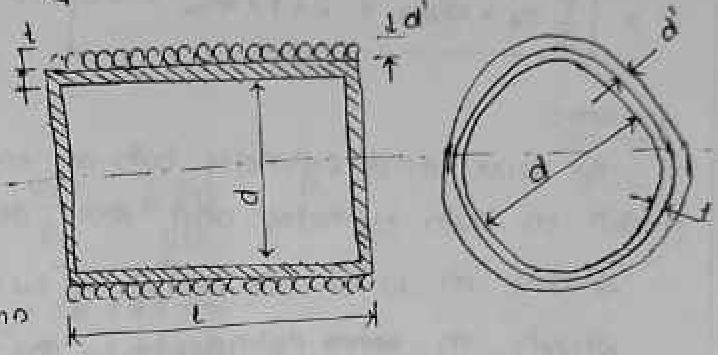
$= \boxed{0.114 \text{ mm}}$

→ volumetric strain $(e_v) = \frac{\Delta V}{V} = \frac{3Pd}{4tE}(1-\mu)$

change in volume $(\Delta V) = \frac{3Pd}{4tE}(1-\mu) \times \frac{\pi}{6} d^3$

$= \frac{3\times2.5\times750}{4\times10\times2\times10^5}(1-0.35)\times\frac{\pi}{6}(750)^3 = \boxed{100955.11 \text{ mm}^3}$

Cylinders are strengthened by closely winding wires under tension. When the cylinder is subjected to internal fluid pressure then the walls of the cylinder are subjected to tensile load. If the cylinder is initially loaded by winding the wires by compressive load then due to the admit of fluid first the tensile load developed compensate the compressive load then after compensation tensile load develops. Hence the load carrying capacity of cylindrical walls increases. This technique of strengthening of cylinders is initially adopted for those materials which are not very strong in tension (cast Iron) and in such case steel wires are used.



Let d' = Diameter of steel wire.

d = Diameter of cylinder

t = Thickness of cylinder

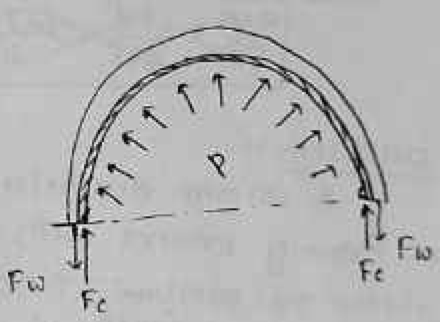
L = Length of cylinder

σ_w = Initial tensile stress on wire

σ_c = Initial compressive stress on cylinder.

<u>when fluid is not admitted:-</u>

Let us consider the half of cylinder.



Force exerted by one turn of cylinder -

$$2\times\frac{\pi}{4}\times(d')^2\times\sigma_w$$

No. of turns of wires at a length $L = L/d'$

Total force exerted by the wire wounding =

$$\frac{L}{d'}\times2\times\frac{\pi}{4}(d')^2\times\sigma_w \quad —— ①$$

Resisting force on the cylinder = $2\times t\times L\times\sigma_c$ —— ②

so for equilibrium eq ① = eq^n ②

$$2\times t\times L\times\sigma_c = \frac{L}{d'}\times2\times\frac{\pi}{4}(d')^2\times\sigma_w$$

$$\Rightarrow \sigma_c = \frac{\frac{\cancel{L}}{\cancel{d'}}\times\cancel{2}\times\frac{\pi}{4}(d')^2\times\sigma_w}{\cancel{2}\times t\times\cancel{L}} = \boxed{\frac{\pi d'\sigma_w}{4t}}$$

<u>when fluid is admitted:-</u>

Let the pressure developed due to fluid pressure only

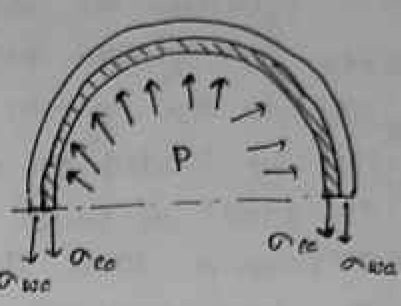
σ_{wa} = In wire

σ_{ca} = In cylinder

Considering the force acting at half portion of cylinder

Force acting on wire & cylinder

$$= 2\times\frac{\pi}{4}\times(d')^2\times\sigma_{wa}\times\frac{L}{d'} + \sigma_{ca}\times 2\times t\times L \quad \text{--- (3)}$$



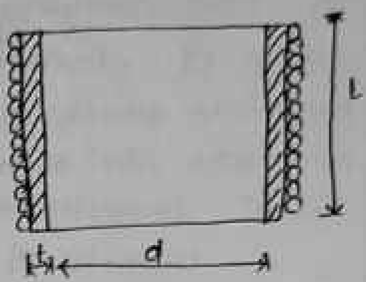
Force due to water pressure

$$= P\times d\times L \quad \text{--- (4)}$$

For equilibrium (3) = (4)

$$= 2\times\frac{\pi}{4}\times(d')^2\times\sigma_{wa}\times\frac{L}{d'} + \sigma_{ca}\times 2\times t\times L = P\times d\times L$$

$$= \boxed{\frac{\pi}{2}d'\times\sigma_{wa} + 2\times t\times\sigma_{ca} = Pd}$$



Strain:-

As wire and cylinder both are rigidly fixed so strain in both cylinder and wire are same.

Strain in wire $(e_w) = \frac{\sigma_w}{E_w}$ [E_w = young's modulus of wire]

Strain in ~~wire~~ cylinder $(e_c) = \frac{\sigma_{ca}}{E} - \mu\frac{\sigma_L}{E} = \cancel{\frac{\mu Pd}{4tE}}$ $= \boxed{\frac{\sigma_{ca}}{E} - \mu\frac{Pd}{4tE}}$

Now $\cancel{\boxed{\frac{\sigma_{wa}}{E_w} = \frac{Pd}{2tE} - \frac{\mu Pd}{4tE}}}$ $\boxed{\frac{\sigma_{wa}}{E_w} = \frac{\sigma_{ca}}{E} - \mu\frac{Pd}{4tE}}$

problem:-

A 250mm diameter CI pipe has metal thickness of 10mm. It is closely wound with 6mm diameter steel wire with an initial strain of 80N/mm². Find the final stresses developed in cylinder and wire when fluid is admitted in to it with a pressure of 3N/mm².

Take $E_c = 100\,kN/mm^2$, $\mu = 0.3$ and $E_s = 200\,kN/mm^2$

Soln

Given data

$d = 250$ mm, $t = 10$mm, $d' = 6$ mm, $\sigma_w = 80\,N/mm^2$

Length of cylinder is not mentioned so let us consider for a length of 6mm for one turn of wire.

Force exerted by the ~~cylinder~~ wire at diameter of cylinder.

$$= \sigma_w\times 2\times\frac{\pi}{4}(d')^2\times\frac{L}{d'}$$

$$= 80\times 2\times\frac{\pi}{4}\times(6)^2\times 6/6$$

$$= \boxed{4521.6\,N}$$

(7)

Initial stress on cylinder -

$$\sigma_c \times 2 \times t \times L = 4521.6$$

$$\Rightarrow \sigma_c = \frac{4521.6}{2 \times t \times L} = \frac{4521.6}{2 \times 10 \times 6} = \boxed{37.68\ N/mm^2}$$

<u>After the fluid admitted</u>:-

$$\sigma_{wo} \times 2 \times \frac{\pi}{4} \times (d')^2 \times \frac{L}{d'} + 2 \times t \times L \times \sigma_{co} = P \times d \times L$$

$$\Rightarrow \sigma_{wo} \times 2 \times \frac{\pi}{4} \times (6)^2 \times \frac{6}{6} + 2 \times 10 \times 6 \times \sigma_{co} = 3 \times 250 \times 6$$

$$\Rightarrow 56.55\, \sigma_{wo} + 120\, \sigma_{co} = 4500$$

$$\Rightarrow \sigma_{wo} + 2.12\, \sigma_{co} = 79.58$$

<u>Now strain equation</u>

$$\frac{\sigma_{wo}}{E_w} = \boxed{\cancel{\frac{Pd}{2tE}} \cancel{\times} \cancel{\frac{\mu Pd}{4tE}}} \quad \frac{\sigma_{co}}{E} + \mu \frac{Pd}{4tE}$$

$$\Rightarrow \frac{\sigma_{wo}}{200 \times 10^3\ N/mm^2} = \frac{\sigma_{co}}{100 \times 10^3\ N/mm^2} + \frac{0.3 \times 3 \times 250}{4 \times 10 \times 100 \times 10^3}$$

$$\Rightarrow \frac{\sigma_{wo}}{2} = \sigma_{co} + 5.625$$

$$\Rightarrow \sigma_{wo} = 2\sigma_{co} + 11.25$$

By solving two equations

$$\sigma_{co} = 22.035\ N/mm^2$$

$$\sigma_{wo} = 32.821\ N/mm^2$$

<u>Final Stresses</u>:-

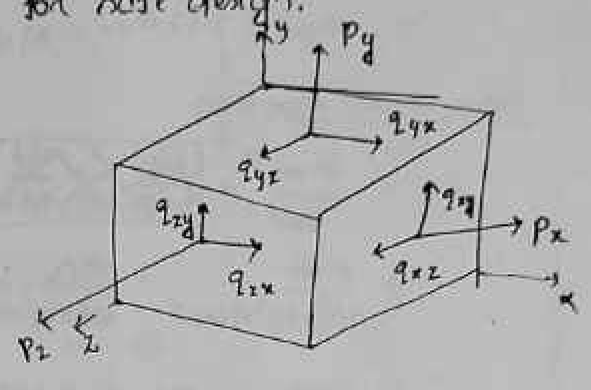
In steel wire $= 80 + 32.821 = 112.82\ N/mm^2$ (Ans)

In cylinder $= -37.5 + 22.035 = -15.465\ N/mm^2$ (Ans)

-: PRINCIPAL STRESS :-

Introduction:-

A structural member may subjected to different types of stresses (normal & shearing) stresses simultaneously. It is therefore necessary to find the region where the effect of these stresses will be critical from the design point of view. when such stresses act at a point in a stressed material, there always exist three orthogonal planes where exists entirely normal stress (with no shear stress). Such planes are called principal planes and such stresses are called principal stresses. One of these principal stresses have greatest value called maximum principal stress and one is having lowest value. The maximum principal should not exceed the permissible value for safe design.



Stresses on an Inclined plane:-

To find the stresses acting at an inclined plane we consider a general plane at an inclined angle 'θ' to the known plane. In that plane we are to find out normal and tangential components

The following three types of stressed condition in an inclined plane is considered.

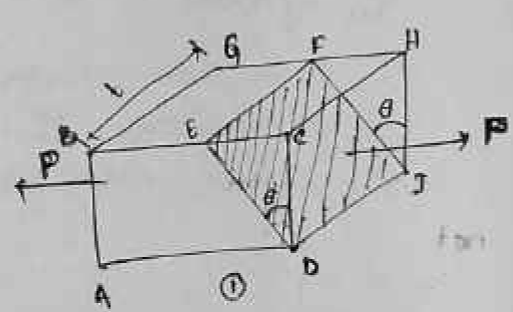
(a) Uniaxial direct stress

(b) Biaxial direct stress

(c) General two dimensional stress system.

Elements subjected to uniaxial direct stress:-

Consider an element subjected to direct uniaxial stress. Let us consider a plane at an angle θ to the plane CDJH i.e DEFJ



Let us analyse at 2D system in fig(2)

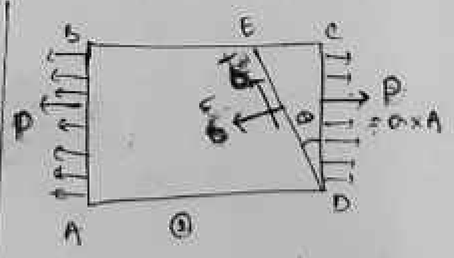
Let the load acting on plane CDHJ = P

Area of plane = A

Stress on plane (σ) = P/A

Cross sectional area of inclined plane (DE)

$$A' = \frac{A}{\cos\theta} = A\sec\theta \qquad \left[\cos\theta = \frac{CD}{DE}\right]$$

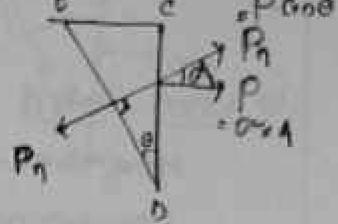


(a)

~~Force acting at inclined plane DE = $\sigma_t \times$ Area(A') = $\sigma_t \times A \sec\theta$~~

Normal ~~stress~~ force at plane DE (P_n) : $P\cos\theta$

$= \sigma \times A\cos\theta$



Normal stress at plane DE = $\frac{P_n}{A'}$

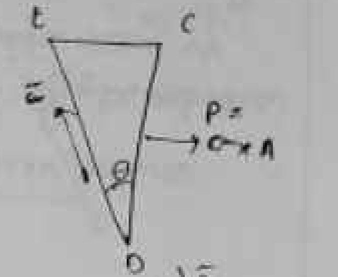
$(\sigma_n) = \frac{\sigma \times A\cos\theta}{A\sec\theta} = \boxed{\sigma\cos^2\theta}$ —— (1)

<u>Tangential stress</u>:- (σ_t or τ)

Tangential force acting at plane DE (P_t) =

$= P\sin\theta = \sigma \times A\sin\theta$

Tangential stress at plane DE = $\frac{P_t}{A'}$



$\tau = \sigma_t = \frac{\sigma \times A\sin\theta}{A\sec\theta} = \sigma . \sin\theta . \cos\theta$

$= \boxed{\frac{\sigma}{2}\sin 2\theta}$ —— (2)

Note:-

From the eq^n (1) & (2)

σ_n is maximum when $\theta = 0$ i.e $\sigma_n = \sigma\cos^2\theta$

$= \sigma\cos^2 0 = \boxed{\sigma}$

σ_n is minimum when $\theta = 90°$ i.e $\boxed{\sigma_n = 0}$

σ_t is maximum when $\sin 2\theta = 1 \Rightarrow 2\theta = 90° \Rightarrow \theta = 45°$

i.e $\boxed{\sigma_t = \sigma/2} \Rightarrow \boxed{\tau = \sigma/2}$

σ_t is minimum when $\sin 2\theta = 0 \Rightarrow 2\theta = 0° \Rightarrow \theta = 0°$

i.e $\boxed{\sigma_t = 0} \Rightarrow \boxed{\tau = 0}$

* σ_n : Normal stress

$\tau = \sigma_t$: ~~[illegible]~~ shear stress

<u>Problem</u>:-

A material has strength in tension, compression and shear as 30 N/mm², 90 N/mm² and 25 N/mm² respectively. If specimen of diameter 20 mm are tested in tension and compression Identify the failure surfaces and loads.

Given,

$\sigma_t = 30$ N/mm², $\sigma_c = 90$ N/mm², $\tau = 25$ N/mm²

(a) <u>when tested under tension</u>:-

when subjected to full tensile strength

stress in tension $\sigma_t = 30$ N/mm²

Corresponding stress in shear (τ) = $\sigma_t/2 = 30/2 = \boxed{15 \text{ N/mm}^2}$

which is less than $25\,N/mm^2$ no failure due to shear will not occur only tensile failure will occur.

corresponding tensile force $(P) = A \times \sigma$

$= \pi/4 \times (20)^2 \times 30 = \boxed{9424.77\,N}$

(b) <u>when tested under compression</u>:-

Maximum compressive stress $(\sigma_c) = 90\,N/mm^2$

Corresponding shear stress $(\tau) = \sigma_c/2 = 90/2 = 45\,N/mm^2$

$> 25\,N/mm^2$

So failure due to shear may occur so the failure surface is at an angle of $45°$ to the plane of axial stress.

As maximum shear stress is $25\,N/mm^2$

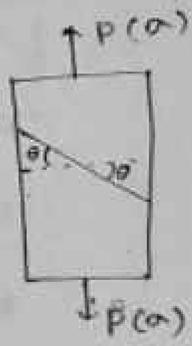
corresponding compressive stress $25 \times 2 = 50\,N/mm^2$

compressive ~~stress~~ Force $(F_c) = A \times \sigma_c$

$= \pi/4 \times (20)^2 \times 50 = \boxed{15707.93\,N}$

problem:-

A circular diameter 25 mm is subjected to an axial force of 20KN as shown in fig . find the stresses on a plane making 30° to the plane of axial stresses and also on the plane which has maximum shear stress.



Soln

Axial stress $(\sigma) = \frac{P}{A} = \frac{20 \times 1000}{\pi/4 \times (25)^2} = 40.737\,N/mm^2$

If $\theta = 30°$

$\sigma_n = \sigma \cos^2\theta = 40.737 \times \cos^2 30 = \boxed{30.558\,N/mm^2}$

shear stress $\sigma_t = \sigma/2 \sin 2\theta = \frac{40.737}{2} \times \sin 2\cdot 30$

$= \boxed{17.643\,N/mm^2}$

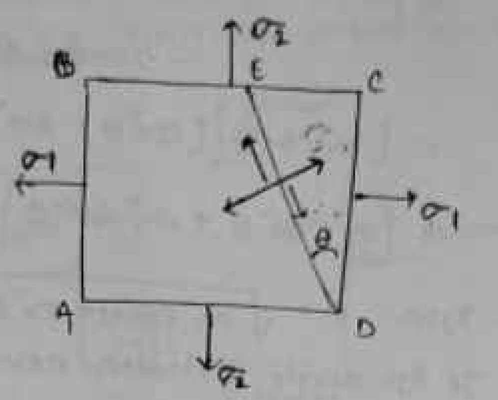
we know maximum shear stress occurs at a plane where $\theta = 45°$

Maximum shear stress $(\sigma_t = \tau) = \frac{\sigma}{2} \sin 2\theta$

$= \frac{40.737}{2} \cdot \sin 90°$

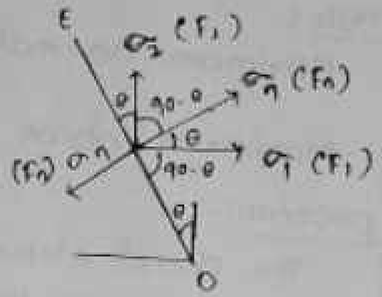
$= \boxed{20.372\,N/mm^2}$

Elements subjected to Biaxial Direct stress:-



Consider an element ABCD having thickness 't' as shown in fig. It is subjected to direct tensile stresses σ_1 & σ_2 as shown in fig. Consider a plane DE at an angle θ inclined to DC.

Now resolving all the forces w.r.t the plane DE



$$F_n = F_1 \cos\theta + F_2 \sin\theta$$

$$\sigma_n \times DE = \sigma_1 \times CD \times \cos\theta + \sigma_2 \times CE \times \sin\theta$$

$$\sigma_n = \sigma_1 \times \frac{CD}{DE} \cos\theta + \sigma_2 \times \frac{CE}{DE} \sin\theta$$

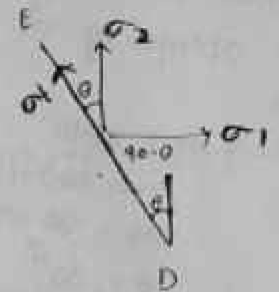
$$= \sigma_1 \times \cos\theta \times \cos\theta + \sigma_2 \times \sin\theta \times \sin\theta$$

$$= \sigma_1 \times \cos^2\theta + \sigma_2 \sin^2\theta$$

$$= \sigma_1\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_2\left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{\sigma_1}{2} + \frac{\sigma_2}{2} + \frac{\sigma_1}{2}\cos 2\theta - \frac{\sigma_2}{2}\cos 2\theta$$

$$\boxed{\sigma_n = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\theta}$$



$$F_t = F_2 \cos\theta - F_1 \cos(90-\theta)$$

$$= F_2 \cos\theta - F_1 \sin\theta$$

$$\sigma_t \times DE = -\sigma_2 \times EC \times \cos\theta + \sigma_1 \times CD \times \sin\theta$$

$$\sigma_t = \frac{-\sigma_2 \times EC \times \cos\theta}{DE} + \frac{\sigma_1 \times CD \times \sin\theta}{DE}$$

$$= -\sigma_2 \sin\theta \cdot \cos\theta + \sigma_1 \cos\theta \cdot \sin\theta$$

$$= (\sigma_1 - \sigma_2) \sin\theta \cdot \cos\theta$$

$$\boxed{\sigma_t = \left(\frac{\sigma_1-\sigma_2}{2}\right)\sin 2\theta}$$

Resultant stress at the plane

$$\sigma = \sqrt{(\sigma_n)^2 + (\sigma_t)^2}$$

$$= \sqrt{(\sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta)^2 + [(\sigma_1-\sigma_2)\sin\theta \cdot \cos\theta]^2}$$

$$= \left[\sigma_1^2 \cos^4\theta + \sigma_2^2 \sin^4\theta + \cancel{2\sigma_1\sigma_2 \sin^2\theta \cos^2\theta} + \sigma_1^2 \sin^2\theta \cdot \cos^2\theta + \sigma_2^2 \sin^2\theta \cdot \cos^2\theta - \cancel{2\sigma_1\sigma_2 \sin^2\theta \cdot \cos^2\theta}\right]^{1/2}$$

$$= \left[\sigma_1^2 \cos^2\theta \left(\cos^2\theta + \sin^2\theta\right) + \sigma_2^2 \sin^2\theta \left(\sin^2\theta + \cos^2\theta\right)\right]^{1/2}$$

$$= \left[\sigma_1^2 \cos^2\theta + \sigma_2^2 \sin^2\theta\right]^{1/2}$$

Then $\sigma = \sqrt{\sigma_1^2 \cos^2\theta + \sigma_2^2 \sin^2\theta}$

If the angle between resultant stress 'σ' and given plane is ϕ then $\tan\phi = \sigma_n/\sigma_t$

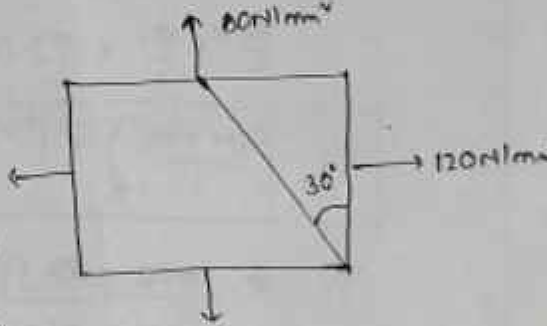
Note:-

Maximum normal stress $(\sigma_n)_{max} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)$ $[\theta = 0°]$

Maximum shear stress $(\sigma_t)_{max} = \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$ $[\theta = 45°]$

problem:-

The direct stresses acting at a point in a strained material are as shown in fig. Find the normal, tangential and the resultant stresses on a plane 30° to the plane of major principal stress. Find the obliquity of the resultant stress also.



Sol:-

Given data

$\sigma_1 = 120\ N/mm^2$

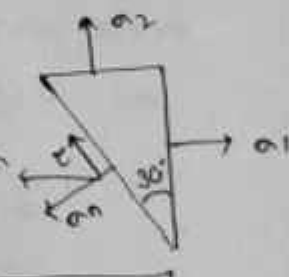
$\sigma_2 = 80\ N/mm^2$

$\theta = 30°$

Normal stress $(\sigma_n) = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta$

$= \frac{120+80}{2} + \frac{120-80}{2}\cos 60°$

$= \boxed{110.00\ N/mm^2}$



Tangential or shear stress $(\sigma_t)(\tau) =$

$\frac{\sigma_1-\sigma_2}{2}\sin 2\theta = \frac{120-80}{2}\sin 60°$

$= \boxed{17.32\ N/mm^2}$

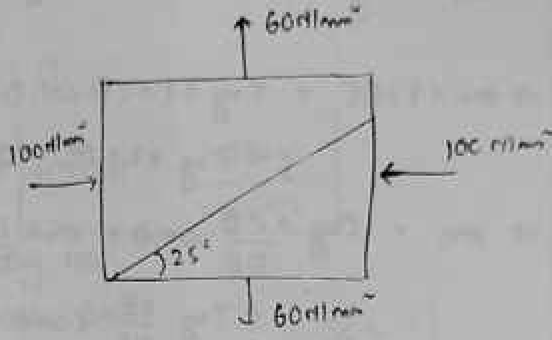
Resultant stress $(\sigma) = \sqrt{(\sigma_n)^2 + (\tau)^2} = \sqrt{(110)^2 + (17.32)^2} = \boxed{111.36\ N/mm^2}$

Angle to inclined plane $\phi = \tan^{-1}\left(\frac{\sigma_n}{\tau}\right) = \tan^{-1}\left(\frac{110}{17.32}\right)$

$= \boxed{81.05°}$

Problem:-

The direct stresses at a point in a strained material are 100 N/mm² compressive and 60 N/mm² tensile as shown in fig. Find the stresses on the plane AC.



Ans

Given data

$\sigma_1 = 60 N/mm^2$

$\sigma_2 = -100 N/mm^2$ (compressive)

$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta$

$= \frac{60-100}{2} + \frac{60-(-100)}{2}\cos 50°$

$= \boxed{31.423 N/mm^2}$

$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta$

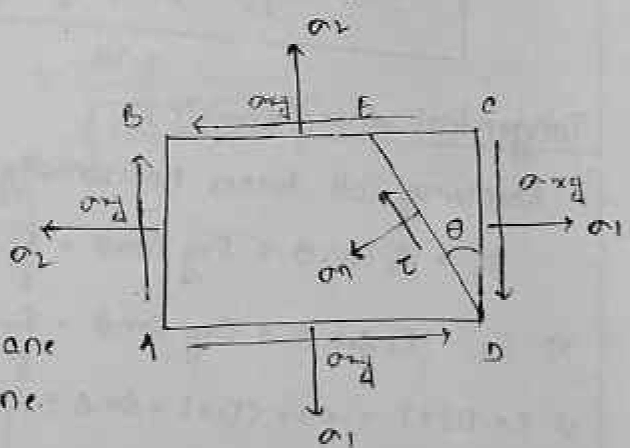
$= \frac{60-(-100)}{2}\sin 50° = 61.284 N/mm^2$

$\sigma = \sqrt{(\sigma_n)^2+(\sigma_t)^2}$ $\sqrt{(31.423)^2+(61.284)^2} = \boxed{68.87 N/mm^2}$

$\theta = \tan^{-1}\frac{\sigma_n}{\sigma_t} = \boxed{27.14°}$

Elements subjected to general two dimensional stress system (Direct stress combined with shear stress):-

Consider an element ABCD subjected to σ_1 & σ_2 normal stresses and τ_{xy} shear stress as shown in fig. Consider a plane at an angle θ.



Let σ_n = Normal stress on inclined plane

τ = Shear stress on inclined plane

Note:-

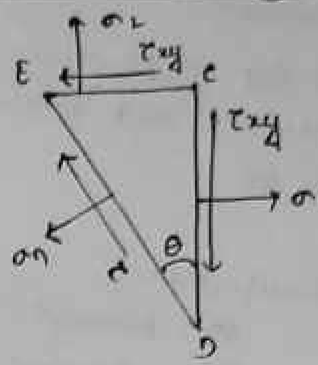
Here the assumed directions for σ_1, σ_2 & τ_{xy} are taken as positive. If the direction changes the stresses have to assumed as negative.

Normal stress (σ_n)

Let the thickness be 't'.

Bringing all the forces to the plane DE

Resolving all the forces vertically



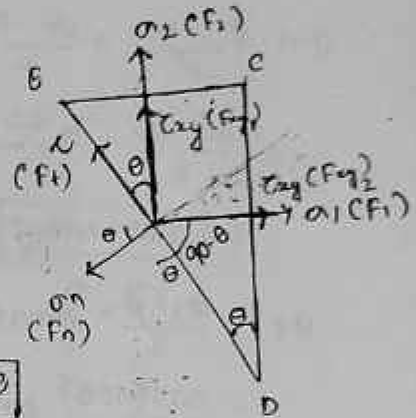
$F_n = F_{xy}\sin\theta + F_2\sin\theta + F_{xy}\sin(90-\theta)$
$\quad + F_1\sin(90-\theta)$

$\Rightarrow \sigma_n \times t \times DE = \tau_{xy}\times t\times CD\sin\theta + \sigma_2\times t\times CE\sin\theta$
$\quad + \tau_{xy}\times t\times CE\cdot\cos\theta + \sigma_1\times t\times CD\cos\theta$

$\Rightarrow \sigma_n = \tau_{xy}\times\frac{CD}{DE}\sin\theta + \sigma_2\times\frac{CE}{DE}\sin\theta$
$\quad + \tau_{xy}\cdot\frac{CE}{DE}\cos\theta + \sigma_1\times\frac{CD}{DE}\cos\theta$

$\Rightarrow \sigma_n = \tau_{xy}\cdot\cos\theta\cdot\sin\theta + \sigma_2\cdot\sin\theta\cdot\sin\theta$
$\quad + \tau_{xy}\cdot\sin\theta\cdot\cos\theta + \sigma_1\cdot\cos\theta\cdot\cos\theta$

$\left[\frac{CD}{DE} = \cos\theta,\ \frac{CE}{DE} = \sin\theta\right]$



$\boxed{\cancel{\Rightarrow \sigma_n = \tau_{xy}\sin^2\theta + \sigma_2\sin^2\theta + \tau_{xy}\cos^2\theta + \sigma_1\cos^2\theta}}$

$\Rightarrow \sigma_n = \sigma_1\cos^2\theta + \sigma_2\sin^2\theta + 2\tau_{xy}\sin\theta\cdot\cos\theta$

$= \frac{\sigma_1}{2}(1+\cos2\theta) + \frac{\sigma_2}{2}(1-\cos2\theta) + \tau_{xy}\sin2\theta$

$$\left[\cos^2\theta = \frac{1+\cos2\theta}{2},\quad \sin^2\theta = \frac{1-\cos2\theta}{2}\right.$$

$$\boxed{\sigma_n = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)\cos2\theta + \tau_{xy}\sin2\theta}$$

Tangential stress $(\sigma_t)(\tau)$:-

Resolving all forces horizontally

$F_t + F_2\cos\theta + F_{xy}\cos\theta = F_1\cos(90-\theta) + F_{xy}\cos(90-\theta)$

$\Rightarrow F_t = F_1\sin\theta + F_{xy}\sin\theta - F_2\cos\theta - F_{xy}\cos\theta$

$\Rightarrow \tau\times DE\times t = \sigma_1\times CD\times t\times\sin\theta + \tau_{xy}\times CE\times t\times\sin\theta - \sigma_2\times CE\times t\times\cos\theta$
$\quad - \tau_{xy}\times CD\times t\times\cos\theta$

$\Rightarrow \tau = \frac{\sigma_1\times CD\cdot\sin\theta}{DE} + \tau_{xy}\times\frac{CE}{DE}\sin\theta - \sigma_2\frac{CE}{DE}\cos\theta - \tau_{xy}\frac{CD}{DE}\cos\theta$

$= \sigma_1\times\cos\theta\cdot\sin\theta + \tau_{xy}\cdot\sin\theta\cdot\sin\theta - \sigma_2\sin\theta\cdot\cos\theta - \tau_{xy}\cos\theta\cdot\cos\theta$

$= \frac{\sigma_1}{2}\sin2\theta - \frac{\sigma_2}{2}\sin2\theta + \tau_{xy}(\sin^2\theta - \cos^2\theta)$

$$\boxed{\sigma_t = \tau = \frac{\sigma_1-\sigma_2}{2}\sin2\theta - \tau_{xy}\cos2\theta}$$

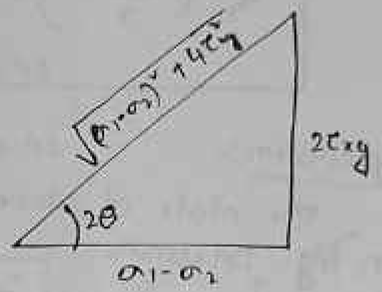
Finding out the principal plane:-

For the principal plane tangential stress is zero so equating σ_t with zero

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau_{xy}\cos 2\theta = 0$$

$$\Rightarrow \frac{\sigma_1-\sigma_2}{2}\sin 2\theta = \tau_{xy}\cos 2\theta \Rightarrow \boxed{\tan 2\theta = \frac{2\tau_{xy}}{\sigma_1-\sigma_2}}$$

There exist two values of 2θ differing by $180°$ satisfying the above equation. Let $2\theta_1$ and $2\theta_2$ be the solution to the equation



$$\sin 2\theta_1 = \frac{2\tau_{xy}}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}} \quad \& \quad \sin 2\theta_2 = \frac{-2\tau_{xy}}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

$$\cos 2\theta_1 = \frac{\sigma_1-\sigma_2}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}} \quad \& \quad \cos 2\theta_2 = \frac{-(\sigma_1-\sigma_2)}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

principal stress-1

$$\sigma_{n1} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\theta_1 + \tau_{xy}\sin 2\theta_1$$

$$= \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)\left(\frac{\sigma_1-\sigma_2}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}\right) + \tau_{xy}\frac{2\tau_{xy}}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

$$= \left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{(\sigma_1-\sigma_2)^2}{2\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}} + \frac{2\tau_{xy}^2}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

$$= \left(\frac{\sigma_1+\sigma_2}{2}\right) + \frac{1}{2}\frac{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}$$

$$\boxed{\sigma_{n1} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau_{xy}^2}}$$

principal stress-2

$$\sigma_{n2} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\theta_2 + \tau_{xy}\sin 2\theta_2$$

Solving similarly

$$\cancel{\sigma_{n2} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \sqrt{\left(\frac{\sigma_1+\sigma_2}{2}\right)^2}}$$

$$\sigma_{n2} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \left(\frac{\sigma_1-\sigma_2}{2}\right)\frac{-(\sigma_1-\sigma_2)}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}} + \tau_{xy}\left(\frac{-2\tau_{xy}}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}\right)$$

$$= \left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{(\sigma_1-\sigma_2)^2}{2\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}} - \frac{2\tau_{xy}^2}{\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

$$= \left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}{2\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}}$$

$$= \left(\frac{\sigma_1+\sigma_2}{2}\right) - \frac{1}{2}\sqrt{(\sigma_1-\sigma_2)^2+4\tau_{xy}^2}$$

$$\boxed{\sigma_{n_2} = \left(\frac{\sigma_1+\sigma_2}{2}\right) - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau_{xy}^2}}$$

Maximum shear stress is derived in the page → 70

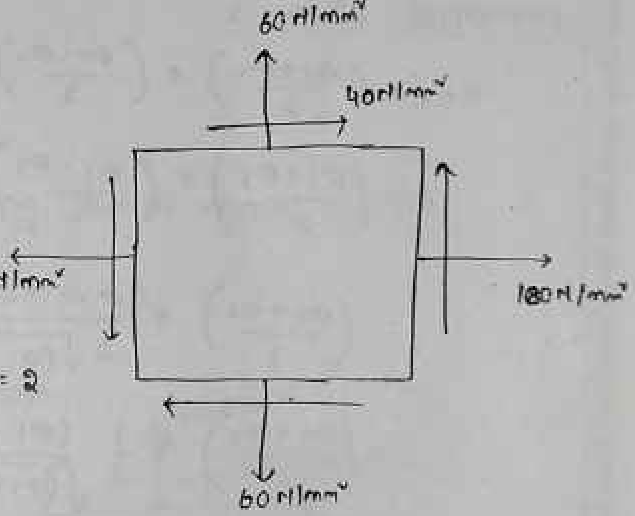
problem:-

The state of stress at a point in a strained material is as shown in fig. Determine

(i) The direction of the principal planes.

(ii) The magnitude of principal stresses

(iii) The magnitude of the maximum shear stress and its direction, indicate all the above planes by sketch.



Soln

Given data,

$\sigma_1 = 100 N/mm^2$, $\sigma_2 = 60 N/mm^2$,

$\tau_{xy} = 40 N/mm^2$

(i) The direction of the principal planes:-

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_1-\sigma_2} = \frac{2\times 40}{100-60} = \frac{80}{40} = 2$$

$$2\theta = \tan^{-1}(2) = 63.43$$

$$\theta = 31.71$$

$$\theta_1 = 31.71, \quad \theta_2 = \theta_1 + 90 = 121.71$$

(ii) principal stresses:-

$$\sigma_{n1} = \left(\frac{\sigma_1+\sigma_2}{2}\right) + \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau_{xy}^2}$$

$$= \left(\frac{100+60}{2}\right) + \sqrt{\left(\frac{100-60}{2}\right)^2+(40)^2}$$

$$= 80 + \sqrt{400+1600} = 80 + 44.72,$$

$$= \boxed{124.72 N/mm^2}$$

$$\sigma_{n2} = \left(\frac{\sigma_1+\sigma_2}{2}\right) - \sqrt{\left(\frac{\sigma_1-\sigma_2}{2}\right)^2+\tau_{xy}^2}$$

$$= \left(\frac{100+60}{2}\right) - \sqrt{\left(\frac{100-60}{2}\right)^2 + (40)^2}$$

$$= 80 - 44.72 = \boxed{35.27\ N/mm^2}$$

(iii) Maximum shear stress:

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_1 - \sigma_2} = \frac{2\times 40}{100-60} = 2$$

$$2\theta = 63.43 \qquad \boxed{\theta = 31.71}$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} \sin 63.43 - \tau_{xy} \cos 63.43$$

$$= \frac{100-60}{2} \sin 63.43 - 40 \cdot \cos 63.43$$

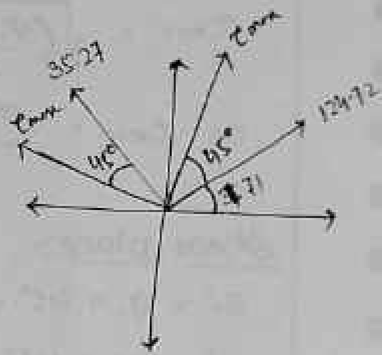
$$= 0$$

$$\tau_{max} = \frac{\sigma_{n1} - \sigma_{n2}}{2} = \frac{124.72 - 35.27}{2} = \boxed{44.725\ N/mm^2}$$

Shear plane:-

$$\theta_1' = \theta_1 + 45^\circ = 31.71 + 45 = \boxed{76.71^\circ}$$

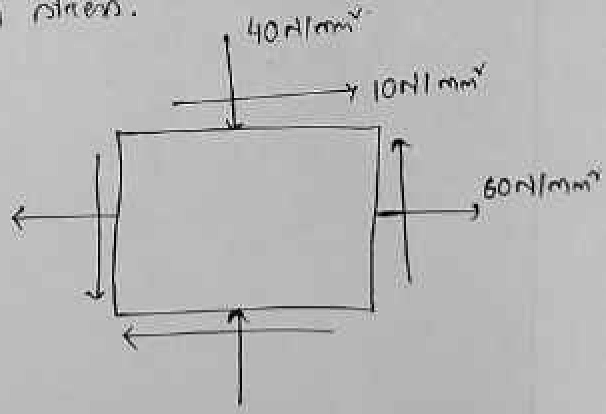
$$\theta_2' = \theta_2 + 45^\circ = 121.71 + 45 = \boxed{166.71^\circ}$$



problem:-

In a strained material the forces acting are as shown in fig. Determine

- → principal planes & principal stresses
- → Maximum shear stresses and planes
- → Resultant stresses in maximum shear stress.



Sol^n

Given data

$\sigma_1 = 60\ N/mm^2$, $\sigma_y = -40\ N/mm^2$

$\tau_{xy}\ 10\ N/mm^2$

principal plane:-

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_1 - \sigma_2} = \frac{2\times 10}{60 - -40}$$

$$= \frac{20}{100} = 11.31^\circ$$

$$\boxed{2\theta_1 = 11.31^\circ,\ 2\theta_2 = 191.31^\circ}$$

$$\boxed{\theta_1 = 5.65^\circ,\ \theta_2 = 95.65^\circ}$$

principal stresses:-

$$\sigma_{n1} = \left(\frac{\sigma_1 + \sigma_2}{2}\right) + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{xy}^2}$$

$$= \left(\frac{60 - 40}{2}\right) + \sqrt{\left(\frac{60 - (-40)}{2}\right)^2 + (10)^2}$$

$$= 10 + 50.99 = \boxed{60.99\ N/mm^2}$$

$$\sigma_{n2} = \left(\frac{\sigma_1 + \sigma_2}{2}\right) - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{xy}^2}$$

$$= \left(\frac{60 - 40}{2}\right) - \sqrt{\left(\frac{60 - (-40)}{2}\right)^2 + (10)^2} = 10 - 50.99 = \boxed{-40.99\ N/mm^2}$$

(comp)

Maximum shear stress:-

$$\tau_{max} = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{60 - -40}{2}\right)^2 + (10)^2} = \boxed{50.99\ N/mm^2}$$

$$\text{or } \tau_{max} = \frac{\sigma_{n1} - \sigma_{n2}}{2} = \frac{60.99 - (-40.99)}{2} = \boxed{50.99\ N/mm^2}$$

Shear plane:-

$$\theta_1' = \theta_1 + 45° = 5.655 + 45 = \boxed{50.655°}$$

$$\theta_2' = \theta_2 + 45° = 95.65 + 45 = \boxed{160.655°}$$

Maximum shear stress:-

As two principal stresses are σ_{n1} & σ_{n2} then the plane of max shear will always lie $\theta_1 + 45°$ & $\theta_2 + 45°$ ($\theta_1 + 135°$) with the reference plane

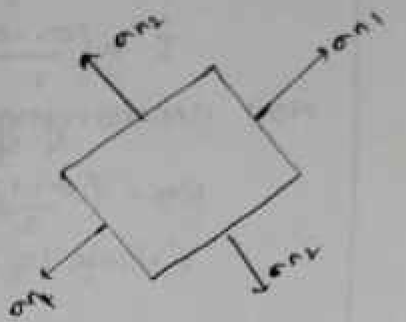
Maximum shear stress

$$\tau_{max} = \frac{\sigma_{n1} - \sigma_{n2}}{2} \sin 2\theta - \tau \cos 2\theta$$

at 45°

$$\tau_{max} = \frac{\sigma_{n1} - \sigma_{n2}}{2} \sin 90 - \tau \times \cos 90$$

$$\therefore \boxed{\tau_{max} = \frac{\sigma_{n1} - \sigma_{n2}}{2}}$$



we can use a generalised formula

$$\boxed{\tau_{max} = \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau_{xy}^2}}$$

Mohr's Circle for stresses:-

Mohr's circle is the graphical representation of stress system. It was proposed by German Civil engineer Otto Mohr in a developed technique

we know that,

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = \left(\frac{\sigma_1-\sigma_2}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Now rearranging the eq^n

$$\sigma_n - \left(\frac{\sigma_1+\sigma_2}{2}\right) = \left(\frac{\sigma_1-\sigma_2}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = \left(\frac{\sigma_1-\sigma_2}{2}\right)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Squaring and adding both equations

$$\left(\sigma_n - \left(\frac{\sigma_1+\sigma_2}{2}\right)\right)^2 + \tau^2 = \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + (\tau_{xy})^2$$

This is equivalent to the equation of circle

$$(x-h)^2 + (y-k)^2 = R^2$$

where $h = \left(\frac{\sigma_1+\sigma_2}{2}\right)$, $k = 0$, $R = \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + (\tau_{xy})^2$

So this system can be represented by a circle.

Procedure for drawing Mohr's circle:-

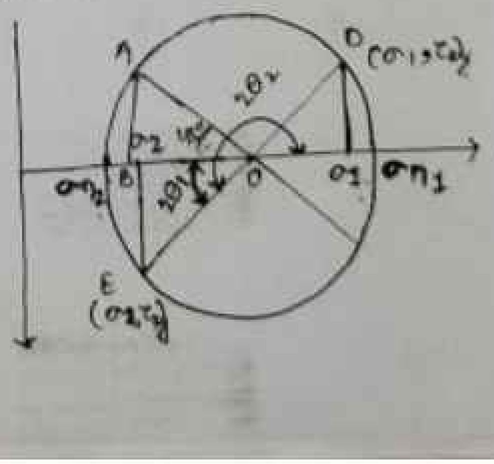
→ consider a co-ordinate system by considering normal stress in x-axis (+ve right side) and shear stress in y-axis (+ve toward down side).

→ Find the points $D(\sigma_1, \tau_{xy})$ and $E(\sigma_2, -\tau_{xy})$ on the co-ordinate system.

→ Join the identified points D&E and locate the point where it is meeting with x-axis.

→ By considering OD as radius draw a circle.

→ The circle is called as Mohr's circle.



Observations:-

σ_{n1} = Radius of circle $+\left(\frac{\sigma_1+\sigma_2}{2}\right)$

σ_{n2} = $\left(\frac{\sigma_1+\sigma_2}{2}\right)$ - Radius of circle

$\Delta\, OD\sigma_1$, OD = Radius of circle

$= \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 + \tau_{xy}^2$

$2\theta_1$ = Angle of σ_{n1}

$2\theta_2$ = Angle of σ_{n2}

$2\theta_1+90^\circ$ = Angle of maximum shear stress 1

$2\theta_2+90^\circ = 2\theta_1+180^\circ$ = Angle of maximum shear stress 2.

AB = Maximum shear stress (It is radius of circle)

Note:-

→ Consider a proper scale (eg $10N/mm^2 = 1cm$) to draw the circle.

→ For shear stress (τ_{xy}) downward direction is considered as -ve. because angle 2θ is considered as +ve when counterclockwise.

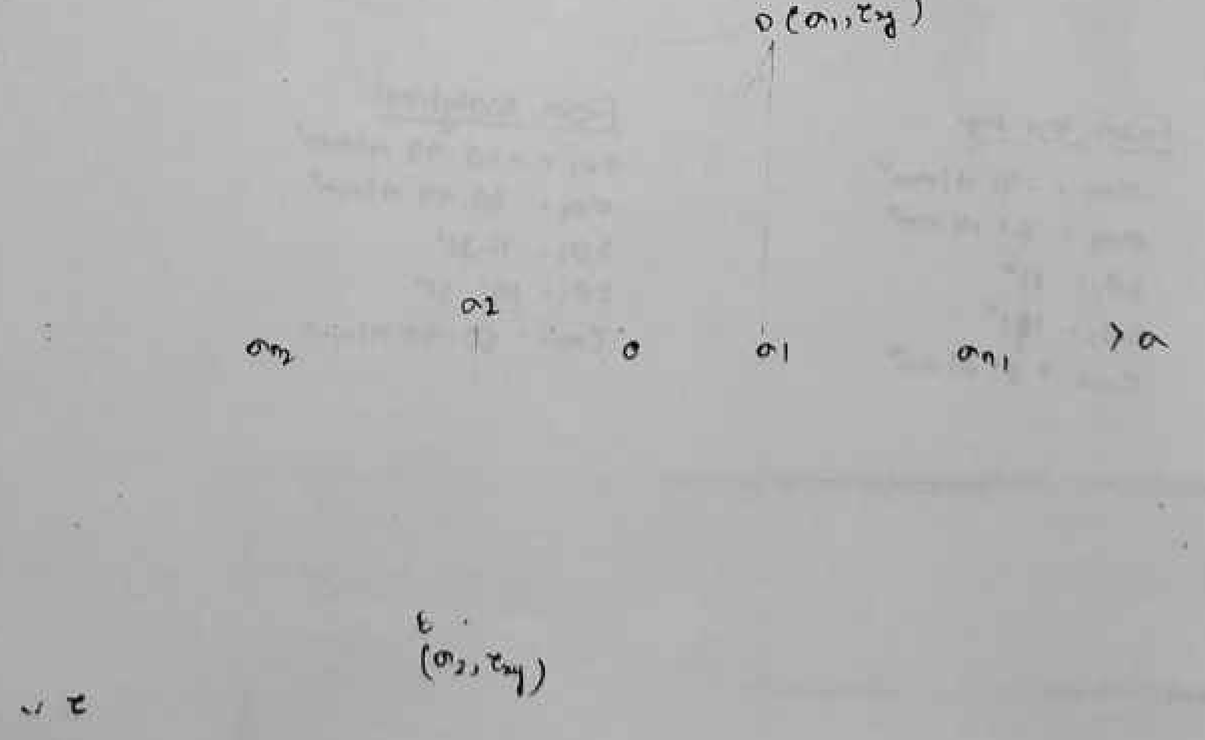
→ Stresses at any plane at any angle can be found out by drawing a line at an angle w.r.t the line DE

→ The corresponding x-axis value will give the normal stress and the y-axis value will give shear stress.

Problem:-

By considering the datas of the problem solved for analytical approach solve for Mohr's circle.

(2) Given data $\sigma_1 = 100N/mm^2$, $\sigma_2 = 60N/mm^2$, $\tau_{xy} = 40N/mm^2$



From the fig,

$\sigma_{n1} = 126\ N/mm^2$

$\sigma_{n2} = 35\ N/mm^2$

$2\theta_1 = 63^\circ$

$2\theta_2 = 243^\circ$

$\tau_{max} = 45\ N/mm^2$

From Analytical,

$\sigma_{n1} = 124.72\ N/mm^2$

$\sigma_{n2} = 35.27\ N/mm^2$

$2\theta_1 = 63.43^\circ$

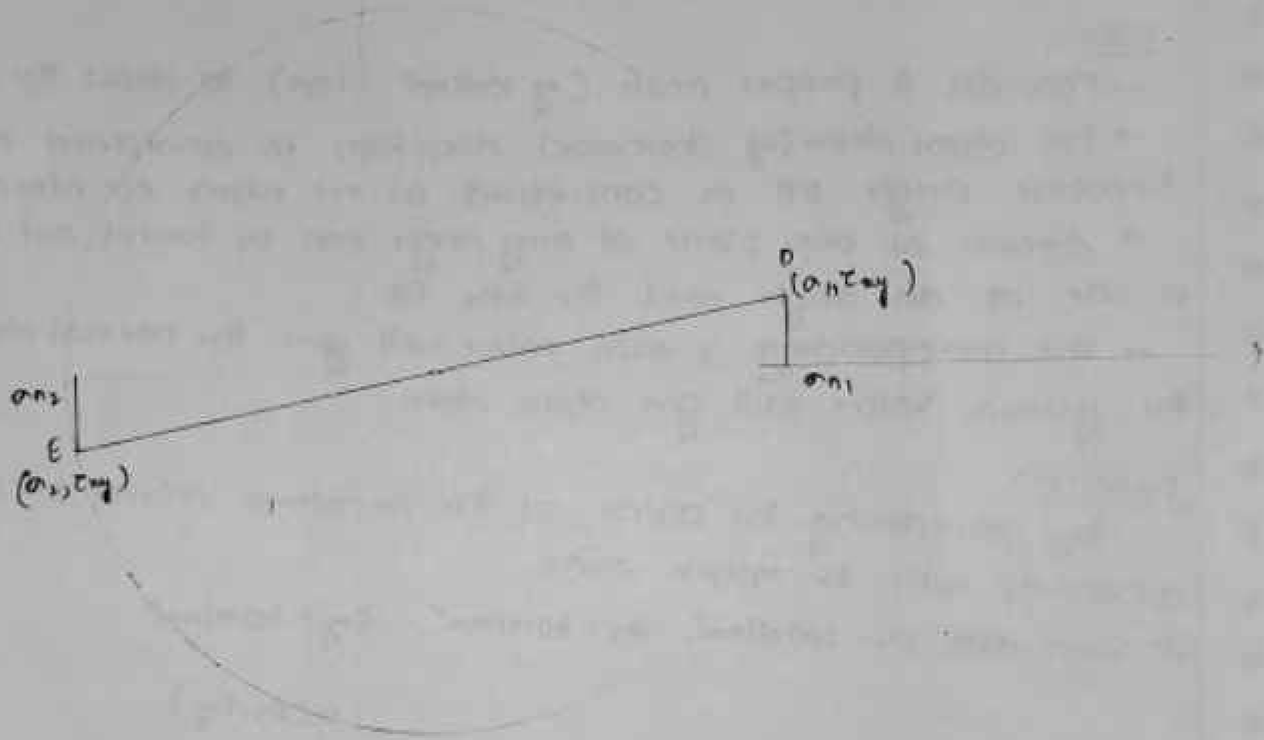
$2\theta_2 = 243.43^\circ$

$\tau_{max} = 44.725\ N/mm^2$

problem - 2

Given data,

$\sigma_1 = 60\ N/mm^2$, $\sigma_2 = -40\ N/mm^2$, $\tau_{xy} = 10\ N/mm^2$



From the fig.

$\sigma_{n1} = -41\ N/mm^2$

$\sigma_{n2} = 61\ N/mm^2$

$2\theta_1 = 11^\circ$

$2\theta_2 = 191^\circ$

$\tau_{max} = 51\ N/mm^2$

From Analytical

$\sigma_{n1} = -40.99\ N/mm^2$

$\sigma_{n2} = 60.99\ N/mm^2$

$2\theta_1 = 11.31^\circ$

$2\theta_2 = 191.31^\circ$

$\tau_{max} = 50.99\ N/mm^2$

The plane on which shearing strain is zero is called principal plane. Similarly to finding principal stresses, we can find principal strains in stressed material, if we know direct strains and shearing strains on a plane that makes an angle w.r.t cartesian direction x and y. For this first we have to find out normal, tangential and shearing strains on a plane that makes angle 'θ' w.r.t the plane of σx stress.

Module :- 2

-: Shear force and Bending Moment Diagram :-

Introduction:-

→ Statically determinate beam:- A beam is said to be statically determinate if its reaction components can be determined by using equations of static equilibrium only.

i.e. $\Sigma F = 0,\ \Sigma M = 0,\ \Sigma W.D = 0$

→ Types of beams:-

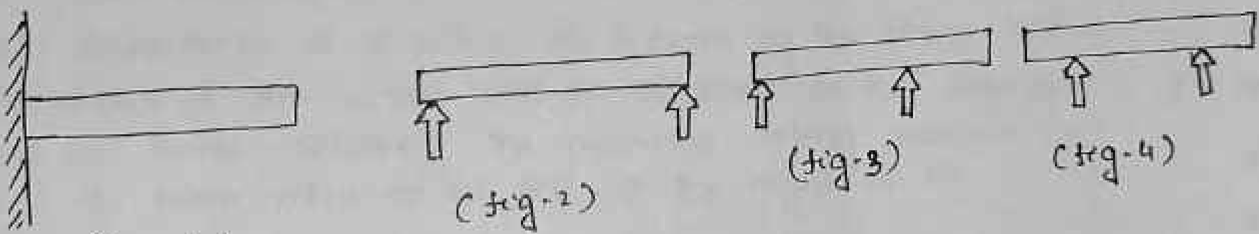
(a) Cantilever Beam:- This type of beam is fixed at one end and free at the other end. (fig-1)

(b) Simply supported Beam:- when the ends of the beams are made to freely rest on supports is called simply supported beam. (fig-2)

(c) Overhanging Beam:- The beam which is having free ends after the support is called overhanging beam

→ Single side overhang (fig-3)

→ Double side overhang (fig-4)



(fig-1) (fig-2) (fig-3) (fig-4)

Note:-

Beam:- Beams are the horizontal members that support vertical loads (tension, compression) ~~or~~, moments and bending.

Columns:- Columns are the vertical members that support axial and lateral loads (compression) & moments.

→ Types of loads usually beams are subjected in transverse direction!

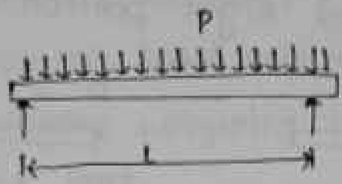
(a) Concentrated load:- If the load is acting at a point. It is also called as point load. (fig-4)

(b) uniformly distributed load:- If the load value is same throughout the beam called UDL. Its value is defined as load acting per unit length × length of the beam. It is supposed to act at the centre of the loaded person. (fig-5(a), 5(b))

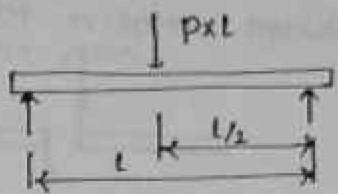
(c) Uniformly varying load:- If the load value varies uniformly through a length is called UVL. Its value is numerically equal to the area under the load and it is supposed to act at the geometric centre of the area. (fig- 6(a), 6(b))

(d) Externally applied moment:- If any moment is acting at any point in the beam directly it is called externally applied moment (fig-7)

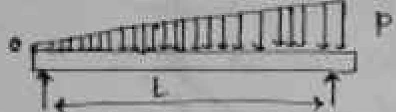
(fig - 4)



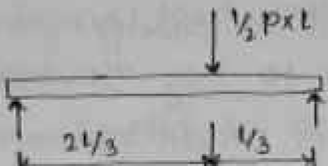
(fig - 5(a))



(fig - 5(b))



(fig - 6(a))



(fig - 6(b))

(fig - 7)

fig 5(b) :- conversion of UDL in to point load.

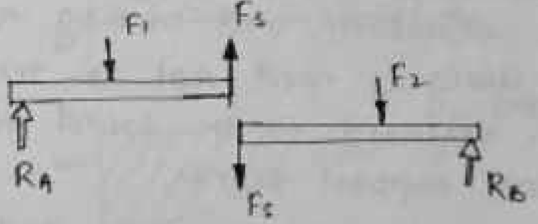
fig 6(b) :- conversion of UVL in to point load.

Shearforce and Bending Moment:-

Shearforce:- Shearforce at a section in a beam is the force that is trying to shear off the section and is obtained as the algebraic sum of all the forces including the reactions acting normal to the axis of the beam either to the left or the right of the section.

Shear force (F_s) = $R_A - F_1$

$= R_B - F_2$

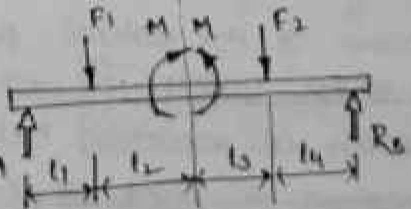


Bending moment:- Bending moment at a section in a beam is the moment that is trying to bend it and is obtained as the algebraic sum of the moments about the section of all the forces (including the reaction acting on the beam either to the left or to the of the section)

Bending moment (M) =

$R_A(l_1+l_2) + F_1 \times l_2$

$= R_B(l_3+l_4) + F_2 l_3$



Sign Convention:-

The shear force and bending moments are the vector quantities and the following sign conventions are generally used.

→ The shear force is positive if it tends to move left portion upward relative to the right portion.

Right side downward +ve
Right side upward -ve (RUN)

(+ve shear force) (-ve shear force)

→ The bending moment is positive if it tries to sag the beam and is negative if it tries to Hog the beam.

Sagging:- when a beam bends in such a type that the top fibres are subjected to compression and bottom fibres are subjected to tension.

(+ve Bending moment)
[Sagging]

Hogging:- when a beam bends in such a way that the bottom fibres are subjected to compression and top fibres are subjected to tension.

[-ve Bending moment]
[Hogging]

Shear force & Bending Moment Diagrams:-

Shear force and bending moment values vary from section to section. A designer need not to find out SF & BM at all the points but it is required to be found out at salient points i.e. at all load and support points.

A diagram in which x-axis represents ~~shear~~ load force and y-axis represents shear force is called shear force Diagram (SFD) and x-axis represents load and y-axis represents bending moment is called bending moment diagram (BMD).

From any SFD & BMD following things can be found.

→ values at the salient points. i.e. maximum, minimum and the points where nature of variation changes.

→ Nature of variation between salient points.

→ Location of point of contraflexure. i.e. the point where bending moment changes its sign obviously value of BM = 0.

SFD and BMD for some standard cases:-

The following standard cases have been considered.

(i) Cantilever subjected to

(a) A concentrated load at free end.

(b) Uniformly distributed load over entire span.

(c) Uniformly varying load over entire span.

(ii) Simply supported beam subjected to

(a) A concentrated load

(b) Uniformly distributed load over entire span.

(c) Uniformly varying load over entire span.

(d) An external moment.

(iii) Overhanging beam subjected to concentrated load at free end.

(i) Cantilever beam:-

(a) Concentrated load at free end:-

Consider a section x-x at a distance of x from point A.

Here $SF_A = +W$ (As right side of section subjected to downward force)

$SF_x = +W$

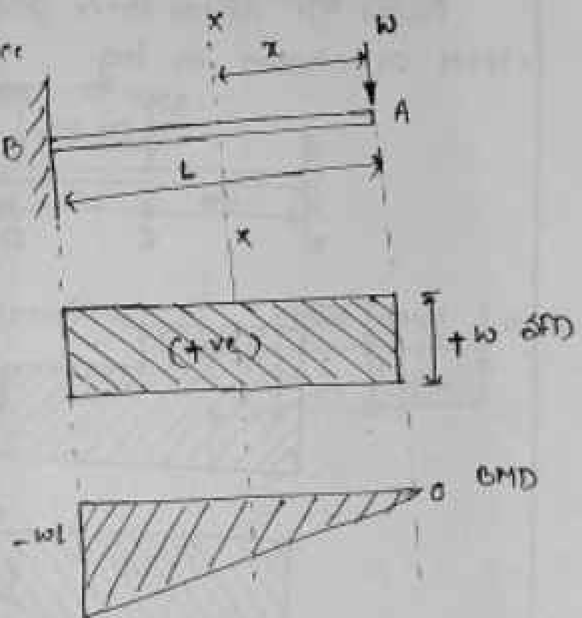
$SF_B = +W$

Bending moment:-

$BM_x = -W \times x$ (Hogging)

$BM_A = 0$

$BM_B = -W \times L$



(b) Uniformly distributed load over entire span:-

Shear force:-

$SF_A = 0$

$SF_x = W \times x$

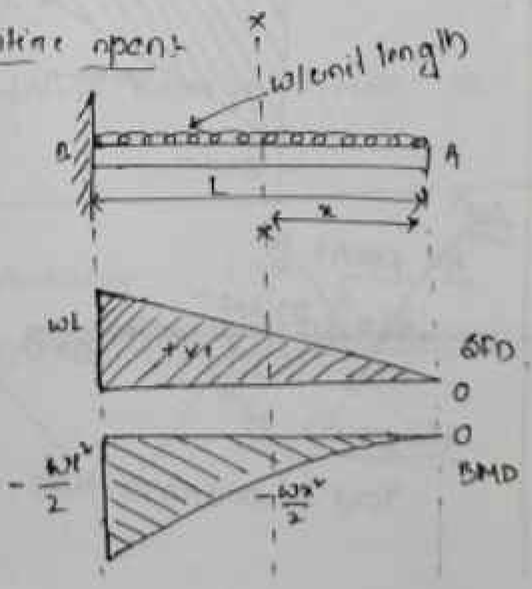
$SF_B = W \times L$

Bending Moment:-

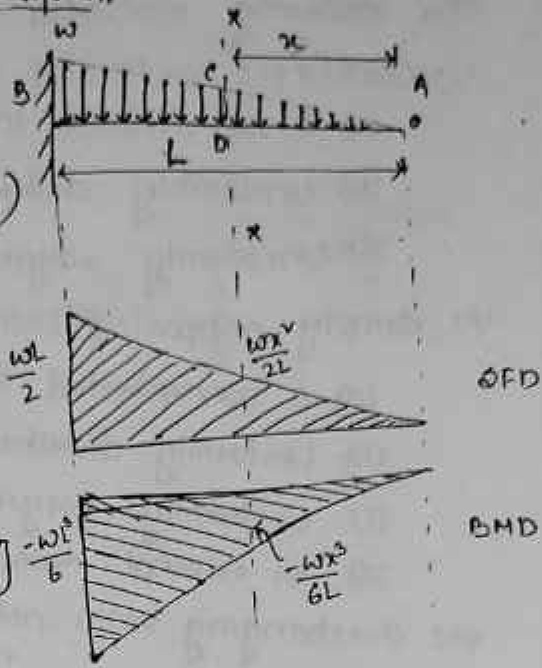
$BM_A = 0$

$BM_x = -W \times x \times x/2 = -Wx^2/2$

$BM_L = -W \times l \times l/2 = -Wl^2/2$



(c) uniformly varying load over entire span:-



Shear force:-

$SF_A = 0$

$SF_x = \frac{1}{2} w \cdot \frac{x}{L} x = \frac{wx^2}{2L}$ (parabolic variation)

(Area of Δ ACD)

$CD = w \times x/L$

$SF_B = \frac{1}{2} w \times L = \frac{wL}{2}$

Bending moment:-

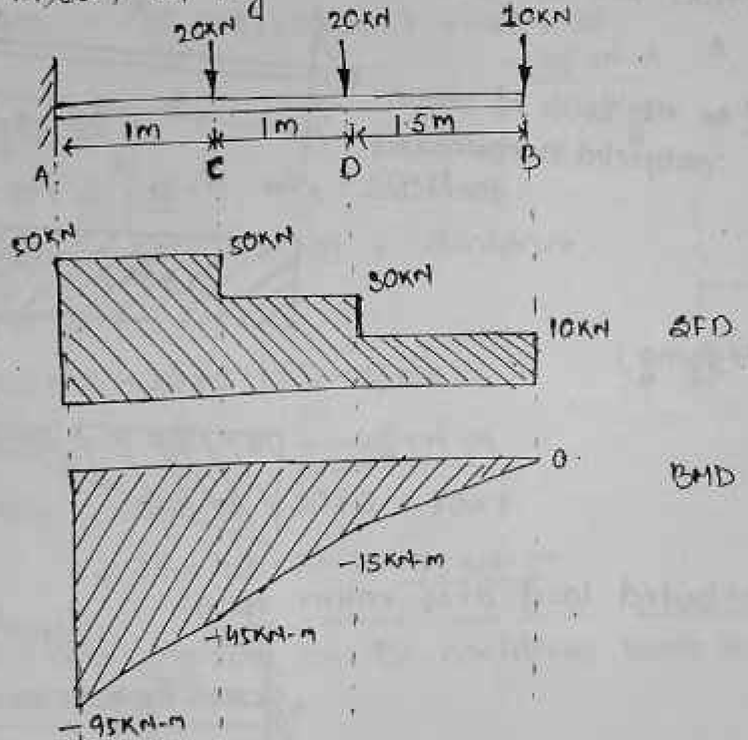
$BM_x = -\frac{1}{2} \frac{wx^2}{L} \times \frac{x}{3} = -\frac{wx^3}{6L}$

$BM_A = 0$ [Distance $= \frac{x}{3}$]

$BM_B = -\frac{wL}{2} \times \frac{L}{3} = -\frac{wL^2}{6}$

problem:-

Draw the shear force and Bending moment diagrams for the canti-lever as shown in fig.



Soln

At point B

$SF_B = 10$ KN

$BM_B = -10 \times x = -10 \times 0 = 0$ KN-m

At point D

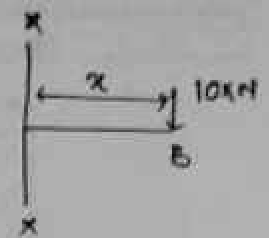
Take a cross section x from free end D.

~~SF_D : 10 + 20 + 30 kN~~
~~BM_D :~~

For the length BD :-

$SF_B = \underline{10\,kN}$ $\quad SF_D = \underline{10\,kN}$

$BM_x = -10x$ [Consider a crosssection xx at a distance x from B]



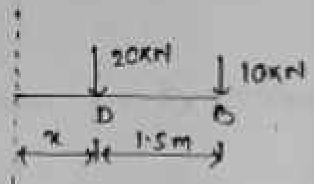
$BM_B = -10\times 0 = \underline{0}$ $[x=0]$

$BM_D = -10\times 1.5 = \underline{-15\,kN\text{-}m}$ $(x = 1.5m)$

For the length CD :-

$SF_D = 10 + 20 = \underline{30\,kN}$, $SF_C = \underline{30\,kN}$

Consider a crosssection xx at x distance from D.



$BM_x = -10(1.5+x) - 20\times x$

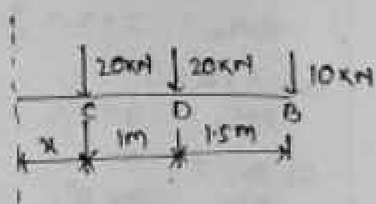
at D, $x=0$ $\quad BM_D = -10\times 1.5 = \underline{-15\,kN\text{-}m}$

at C, $x=1$ $\quad BM_C = -10\,(1.5+1) - 20\times 1 = -25 - 20$
$= \underline{-45\,kN\text{-}m}$

For the length AC :-

$SF_D = 10 + 20 + 20 = \underline{50\,kN}$, $SF_A = \underline{50\,kN}$

Consider a crosssection xx at x distance from the point C.



$BM_x = -10(2.5+x) - 20\times(1+x) - 20\times x$

at C, $x=0$ $\quad BM_C = -25 - 20 = \underline{-45\,kN\text{-}m}$

at A, $x=1$ $\quad BM_A = -10\times 3.5 - 20\times 2 - 20\times 1$
$= -35 - 40 - 20 = \underline{-95\,kN\text{-}m}$

Problem:-

Draw the SF & BM Diagram for the cantilever beam as shown in fig.

Sol:

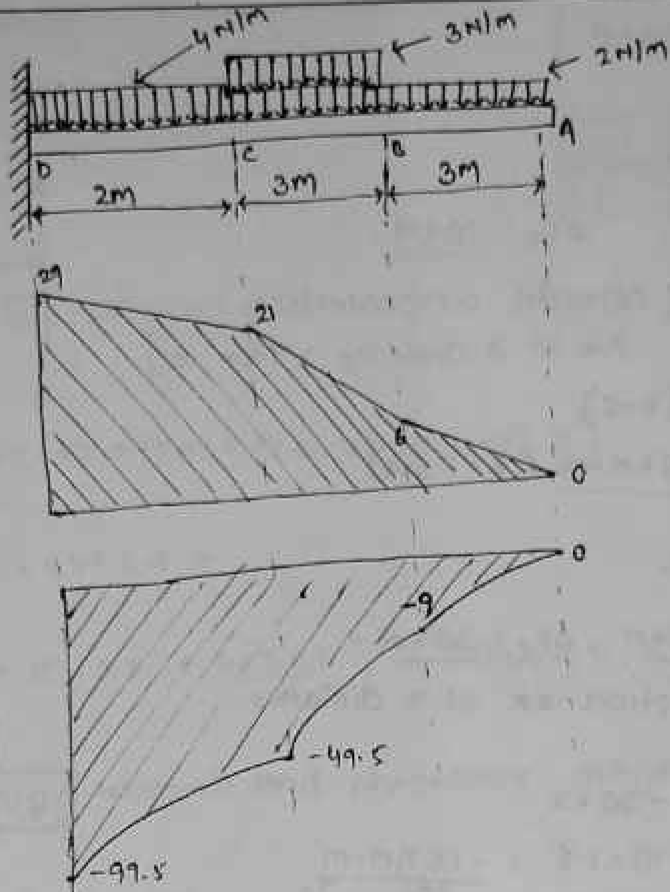
For the length AB

consider a cross section xx at a distance x from the free end.

Shear force

$SF_x = 2\times x\ N$

$SF_A = 2\times 0 = 0N$ $[x=0]$



at B, $x = 3m$

$SF_B = 2 \times 3 = 6N$

Bending moment

$BM_x = 2 \times x \times \frac{x}{2} = -\frac{2x^2}{2}$ (Hogging)

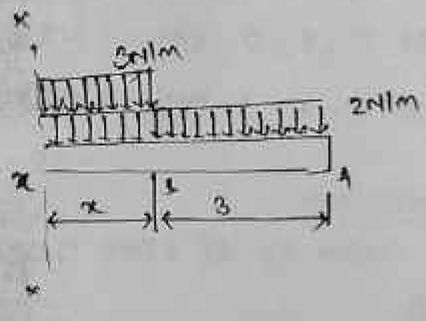
at A, $x = 0$

$BM_A = 2 \times 0^2/2 = -0$ N-m

at B, $x = 3$

$BM_B = 2 \times 3^2/2 = -9$ N-m

For the length BC



Shear force

$SF_x = 2 \times 3 + [(3+2) \times x]$

at B, $x = 0$

$SF_B = 2 \times 3 = 6$ N-m

$SF_C = 2 \times 3 + [(3+2) \times 3]$ $[x = 3]$

$= 6 + 15 = 21$ N-m

Bending moment

$BM_x = -\left[(2 \times 3) \times \left(\frac{3}{2} + x\right) + \left[(3+2) \times x \times \frac{x}{2}\right]\right]$

at B, $x = 0$

$BM_B = -9$ N-m

at C, $x = 3$

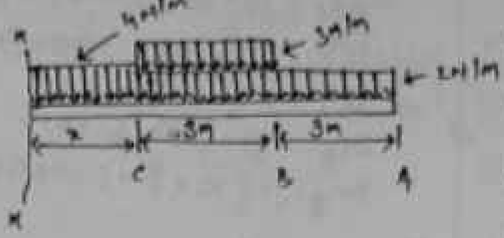
$BM_C = -\left[6 \times 4.5 + 5 \times \frac{3^2}{2}\right] = -49.5$ N-m

For the length CD

Shear force

$SF_x = (2\times3)+(2\times3)+(3\times3)+(4\times x)$

$= 21+4x$



at C, $x=0$

$SF_C = 21+4\times0 = 21N$

$SF_D = 21+4\times2 = 29N \quad (x=2m)$

Bending moment:-

$BM_x = -\left[(2\times3)(4.5+x)+(5\times3)(1.5+x)+(4\times x)\frac{x}{2}\right]$

at C, $x=0$

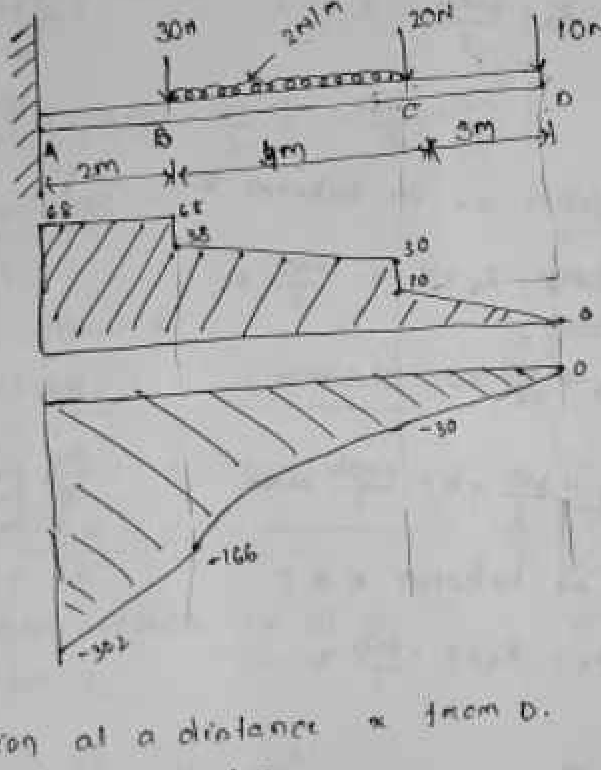
$BM_C = 27+22.5 = -49.5N\text{-}m$

at D, $x=2m$

$BM_D = -\left[6\times6.5+15\times3.5+\frac{4\times2\times2}{2}\right] = -99.5N\text{-}m$

Problem:-

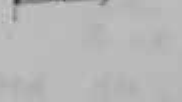
Draw the bending moment and shear force diagram for the cantilever as shown in fig.



CD

consider a section at a distance x from D.

$SF_x = 10N$, $BM_x = -10\times x$ N-m



at D, $x=0$

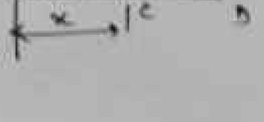
$SF_D = 10N$, $BM_D = 0N\text{-}m$

at C, $x=3m$

$SF_C = 10N$, $BM_C = -10\times3 = -30N\text{-}m$

BC

$SF_x = 10+20+(2\times x)N$, $BM_x = -\left[10\times(x+3)+20\times x + 2\times x\times x/2\right]N\text{-}m$



at C ; $x = 0$

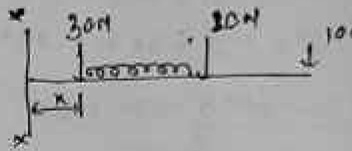
$SF_C = 10 + 20 + (2 \times 0) = 30 \text{ KN}$

$BM_C = -[10 \times 3 + (20 \times 0) + (2 \times 0)] = \boxed{-30 \text{ KN-m}}$

at B, $x = 4m$

$SF_B = 10 + 20 + (2 \times 4) = 38 \text{ KN}$

$BM_B = -[10 \times 7 + (20 \times 4) + (2 \times 4 \times 4/2)] = \boxed{-166 \text{ KN-m}}$



AB

$SF_x = 10 + 20 + 2 \times 4 + 30 \text{ KN}$

$BM_x = -[10 \times (7 + x) + 20(4 + x) + 2 \times 4 \times (4/2 + x) + 30 \times x]$

at B, $x = 0$

$SF_B = 68 \text{ N}$

$BM_B = \boxed{-166 \text{ N-m}}$

at A, $x = 2m$

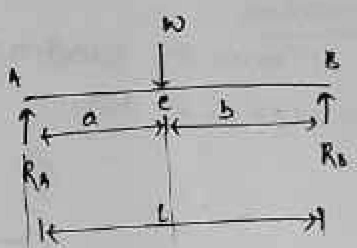
$SF_A = 68 \text{ N}$

$BM_A = -[10 \times 9 + 20 \times 6 + 2 \times 4 \times (2 + 2) + 30 \times 2]$

$= \boxed{-302 \text{ N-m}}$

Simply Supported Beam:-

(a) Subjected to point load:-



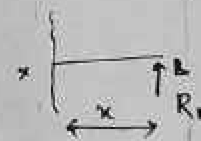
$R_A + R_B = W$

Taking moment about A

$R_B \times l = W \times a \Rightarrow R_B = \frac{W \times a}{l}$

$R_A = \frac{Wb}{l}$

Considering a section xx in between B C



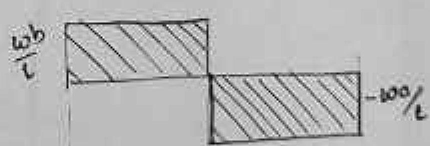
$SF_x = -R_B = -\frac{Wa}{l}$ $\quad BM_x = R_B \times x = \frac{Wa}{l} x$

at B $x = 0$

$SF_B = -\frac{Wa}{l} \text{ N}$ $\quad BM_B = \frac{Wa}{l} \times 0 = 0 \text{ N-mm}$

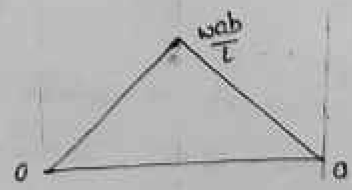
at C $x = b$

$SF_C = -\frac{Wa}{l} \text{ N}$ $\quad BM_C = \frac{Wa}{l} \times b = \frac{Wab}{l} \text{ N-mm}$



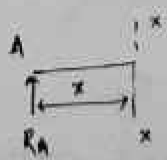
Considering section xx between A & C

$SF_x = R_A = \frac{Wb}{l}$ $\quad BM_x = R_A \times x = \frac{Wb}{l} x$



at A, $x = 0$

$SF_A = \frac{Wb}{l}$ $\quad BM_A = 0$

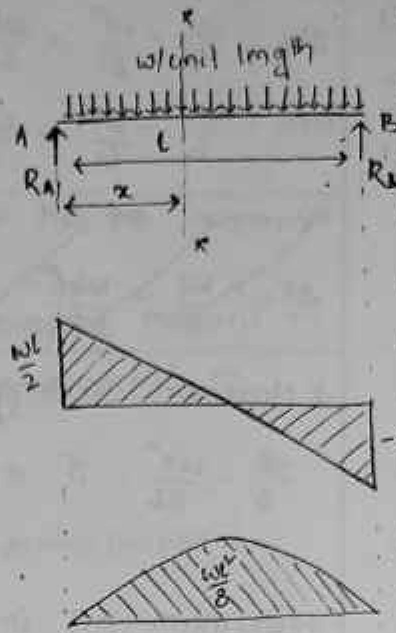


at C, $x = a$

$SF_C = \frac{Wb}{l}$ $\quad BM_C = \frac{Wab}{l}$

2-10 (84)

(b) Subjected to UDL :-



$R_A + R_B = w \times l$

Taking moment about A

$R_B \times l = w \times l \times l/_2$

$R_B = wl/_2$, $R_A = \frac{wl}{2}$

consider a cross section x-x at a distance x from A.

At x

$SF_x = R_A - wx = \frac{wl}{2} - wx$

$BM_x = R_A x - w \cdot x \cdot \frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2}$

at A, $x = 0$

$SF_A = \boxed{wl/_2}$ $BM_A = \boxed{0}$

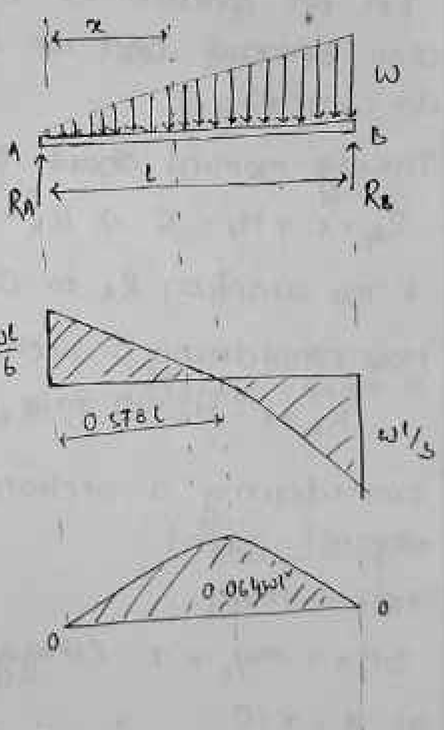
at B, $x = l$

$SF_B = \frac{wl}{2} - wl = \boxed{-\frac{wl}{2}}$, $BM_B = \frac{wl^2}{2} - \frac{wl^2}{2} = \boxed{0}$

at $x = l/_2$

$SF_x = \frac{wl}{2} - \frac{wl}{2} = \boxed{0}$, $BM_x = \frac{wl}{2} \cdot \frac{l}{2} - \frac{wl^2}{8} = \boxed{\frac{wl^2}{8}}$

(c) Subjected to UVL :-



$R_A + R_B = 1/_2 \times w \times l$

Taking moment about A

$R_B \times l = 1/_2 \times w \times l \times 2l/_3$

$R_B = \frac{wl \times l}{3 \times l} = \frac{wl}{3}$

$R_A = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6}$

consider a cross section xx at a distance x from A.

at x

$SF_x = R_A - \frac{1}{2}\frac{w}{l}x \cdot x = \frac{wl}{6} - \frac{wx^2}{2l}$

$BM_x = R_A x - \frac{wx^2}{2l} \cdot \frac{x}{3} = \frac{wl}{6}x - \frac{wx^3}{6l}$

at A, $x = 0$

$SF_A = wl/_6$ $BM_A = 0$

at B, $x = l$

$SF_B = \frac{wl}{6} - \frac{wl^2}{2l} = \frac{wl}{6} - \frac{wl}{2} = \frac{wl-3wl}{6} = \frac{-wl}{3}$

$BM_B = \frac{wl^2}{6} - \frac{wl^2}{6l} = \frac{wl^2}{6} - \frac{wl^2}{6} = 0$

~~Maximum BM will occur at middle point where~~ $x = \frac{2l}{3}$

$\cancel{SF_x = \frac{wl}{6} - \frac{w4l^2}{2\times 3l} = \frac{wl}{6} - \frac{4wl}{6} =}$

* Maximum bending moment will occur at a point where SF is zero

$\frac{wl}{6} - \frac{wx^2}{2l} = 0 \Rightarrow \frac{wl}{6} = \frac{wx^2}{2l} \Rightarrow x^2 = \frac{wl}{6} \times \frac{2l}{w} = \frac{1}{3}l^2$

$\Rightarrow x = \sqrt{1/3}\, l = \boxed{0.578L}$

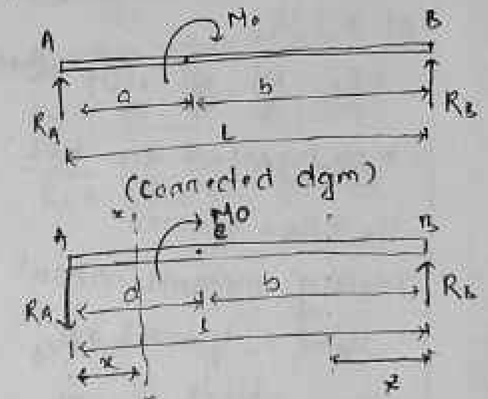
Maximum BM is at $x = 0.578L$

$BM_{max} = \frac{wl}{6}(0.578l) - \frac{w(0.578l)^3}{6l} = (0.096l^2 - 0.032l^2)w$
$= \boxed{0.064wl^2}$

(d) Subjected to Moment:-

Let a moment M_0 acting clockwise direction. As the moment is clockwise so reaction at B is upward.

Let us assume the reaction at A is also upward and we will check for its validity.



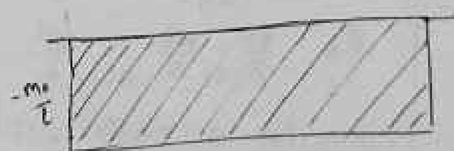
Taking moment about B.

$R_A \times l + M_0 = 0 \Rightarrow R_A = -M_0/l$

* The reaction R_A is acting downward.

Now considering a section

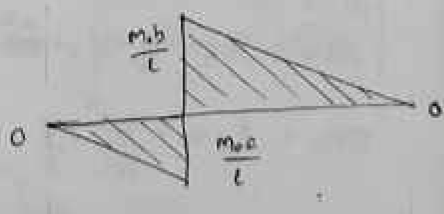
$R_B \times l - M_0 = 0 \Rightarrow R_B = M_0/l$



considering a section xx left to the moment point

$SF_x = -M_0/l$

$BM_x = -M_0/l \times x$ (Hogging)



at A, $x = 0$

$SF_A = \boxed{-M_0/l}$ $\quad BM_A = \frac{M_0}{l}\times 0 = \boxed{0}$

at C, $x = a$

$SF_C = \boxed{-M_0/l}$ $\quad BM_C = \boxed{-\frac{M_0}{l}\times a}$

$SF_x = -M_0/l$, $BM_x = \frac{M_0}{l} \times x$

at B, $x = 0$

$SF_B = \boxed{-M_0/l}$, $BM_B = \boxed{0}$

at C, $x = b$

$SF_C = \boxed{-M_0/l}$, $BM_C = \boxed{\frac{M_0}{l} \times b}$

Imp
Point of contraflexure:-

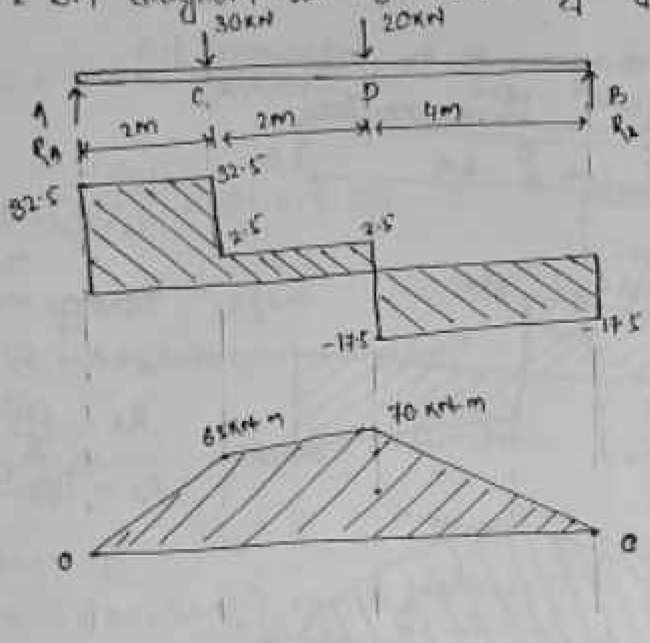
point of contraflexure is a point where the bending moment is changing its sign. At this point bending moment is zero.

Note:-

At the previous section point 'c' is the point of contraflexure.

problem:-

Draw the SF & BM diagram for the following fig.



Note:-

$R_A + R_B = 20 + 30 = 50KN$

Taking moment about A

$R_B \times 8 = 20 \times 4 + 30 \times 2$

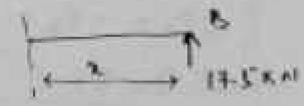
$R_B = \frac{80 + 60}{8} = 17.5$ KN

$R_A = 32.5$ KN

Sol

Let us consider a cross section between B & D at x distance from B.

$SF_x = -17.5$ KN , $BM_x = 17.5 \times x$ KN-m



at B, $x = 0$

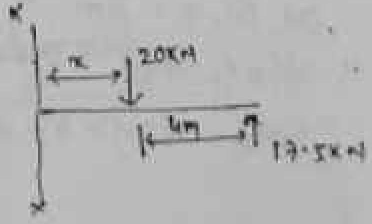
$SF_B = \boxed{-17.5KN}$, $BM_B = \boxed{0}$

at D, $x = 4m$

$SF_D = \boxed{-17.5KN}$, $BM_D = \boxed{70 \text{ KN-m}}$

CD

Consider a section at x distance from 'D'.



$SF_x = -17.5 + 20 = 2.5KN$

$BM_x = 17.5 (4+x) - 20 \times x$

at D, $x = 0$

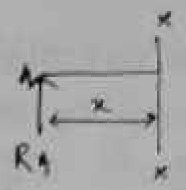
$SF_D = \boxed{2.5\,KN}$, $BM_D = 17.5 \times (4+0) - 20 \times 0 = \boxed{70\,KN\text{-}m}$

at C, $x = 2m$

$SF_C = \boxed{2.5\,KN}$, $BM_C = 17.5 \times (4+2) - 20 \times 2 = \boxed{65\,KN\text{-}m}$

AC

Consider a section xx at x-distance from 'A'



$SF_x = R_A = 32.5\,KN.$

$BM_x = R_A \times x = 32.5 \times x\ KN\text{-}m$

at A, $x = 0$

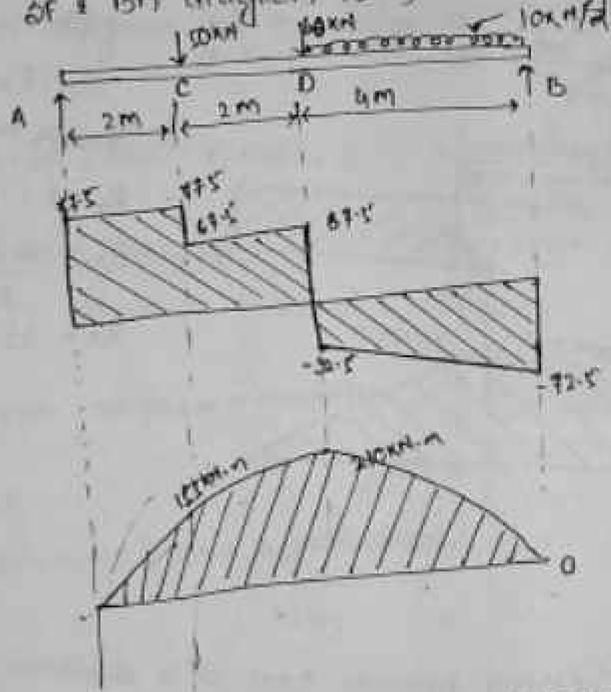
$SF_A = \boxed{32.5\,KN.}$, $BM_A = 32.5 \times 0 = \boxed{0\,KN\text{-}m}$

at C, $x = 2m$

$SF_C = \boxed{32.5\,KN}$, $BM_C = 32.5 \times 2 = \boxed{65\,KN\text{-}m}$

problem :-

Draw the SF & BM diagram to the following fig.



Note :-

$R_A + R_B = 50 + 60 + 10 \times 4$
$= 150\,KN$

Taking moment about B

$R_A \times 8 = 50 \times 6 + 60 \times 4 + 10 \times 4 \times 4/2$

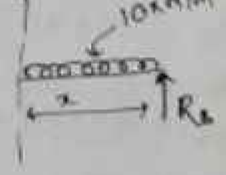
$R_A = \frac{620}{8} = 77.5\,KN.$

$R_B = 150 - 77.5 = 72.5\,KN.$

Sol^n

BD

Consider a section xx from B at a distance 'x'.



$SF_x = -R_B + 10 \times x$, $BM_x = R_B \times x - 10 \times x \times x/2$

at B, $x = 0$

$SF_B = \boxed{-72.5\,KN}$, $BM_B = \boxed{0}$

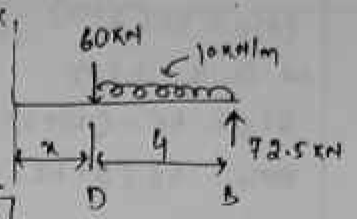
at D, $x = 4m$

$SF_D = -72.5 + 10 \times 4 = \boxed{-32.5\,KN}$

$BM_D = 72.5 \times 4 - 10 \times 4 \times 4/2 = \boxed{210\,KN\text{-}m}$

CD

consider a section xx at a distance 'x' from D.



$SF_x = -72.5 + (10\times4\times4/2) + 60 = 67.5$ KN.

$BM_x = 72.5\times(4+x) - [10\times4\times(4/2+x)] - 60\times x$ KN-m

at D, x = 0

$SF_D = \boxed{67.5 \text{ KN.}}$ $BM_D = 72.5\times4 - [10\times4\times4/2] - 0 = \boxed{210 \text{ KN-m}}$

at C, x = 2

$SF_C = \boxed{67.5 \text{ KN}}$ $BM_C = 72.5(4+2) - [10\times4\times(2+2)] - 60\times2 = \boxed{155 \text{ KN-m}}$

AC

Consider a distance x from A

$SF_x = 77.5$ KN , $BM_x = 77.5\times x$ KN-m

at A, x = 0

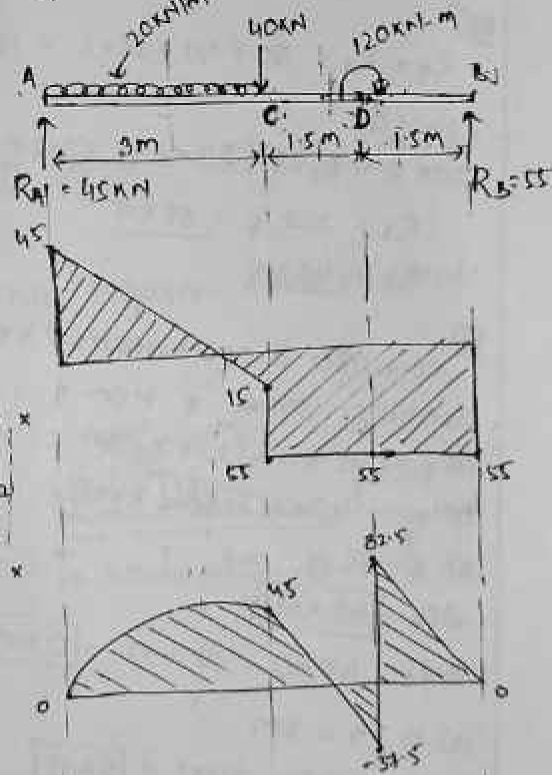
$SF_A = \boxed{77.5 \text{ KN}}$ $BM_A = \boxed{0 \text{ KN-m}}$

at C, x = 2m

$SF_C = \boxed{77.5 \text{ KN}}$ $BM_C = 77.5\times2 = \boxed{155 \text{ KN-m}}$

problem:-

Draw the SF & BM diagram for the following condition.



Sol^n $\Sigma M_B = 0$

$R_A\times6 - 20\times3(1.5+3) - 40\times3 + 120 = 0$

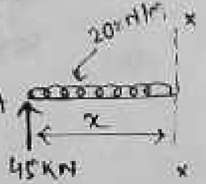
$R_A = 45$ KN.

$R_B = (20\times3) + 40 - 45 = \underline{55 \text{ KN}}$

* 120KN-m in clockwise moment

AC

Consider a section xx ~~from~~ a distance x from 'A'.



$SF_x = 45 - 20\times x$ KN

$BM_x = 45\times x - (20\times x\times x/2)$ KN-m

at A, x = 0

$SF_A = \boxed{45 \text{ KN}}$

$BM_A = \boxed{0 \text{ KN-m}}$

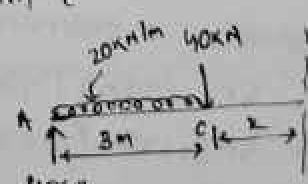
at C, x = 3m

$SF_C = 45 - 20\times3 = 45 - 60 = \boxed{-15 \text{ KN}}$

$BM_C = 45\times3 - (20\times3\times3/2) = \boxed{45 \text{ KN-m}}$

CD

Consider a section xx at a distance x from 'C'



$SF_x = 45 - (20\times3) - 40$ KN

$BM_x = 45(3+x) - [20\times3\times(3/2+x)] - 40\times x$ KN-m

at C, x = 0

$SF_C = 45 - 100 = \boxed{-55 \text{ KN}}$

$BM_C = 45(3+0) - [20\times3\times1.5] - 40\times0 = \boxed{45\ kN\text{-}m}$

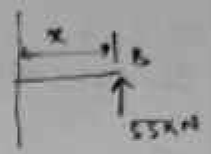
at D, $x = 1.5\,m$

$SF_D = 45 - (20\times3) - 40 = \boxed{-55\,kN}$

$BM_D = 45(3+1.5) - [20\times3\times(1.5+1.5)] - 40\times1.5 = \boxed{-37.5\,kN\text{-}m}$

BD

Consider a section xx at a distance x from B.



$SF_x = -55\,kN$

$BM_x = 55\times x\ kN\text{-}m$

at B, $x = 0$

$SF_B = \boxed{-55\,kN}$, $BM_B = \boxed{0\ kN\text{-}m}$

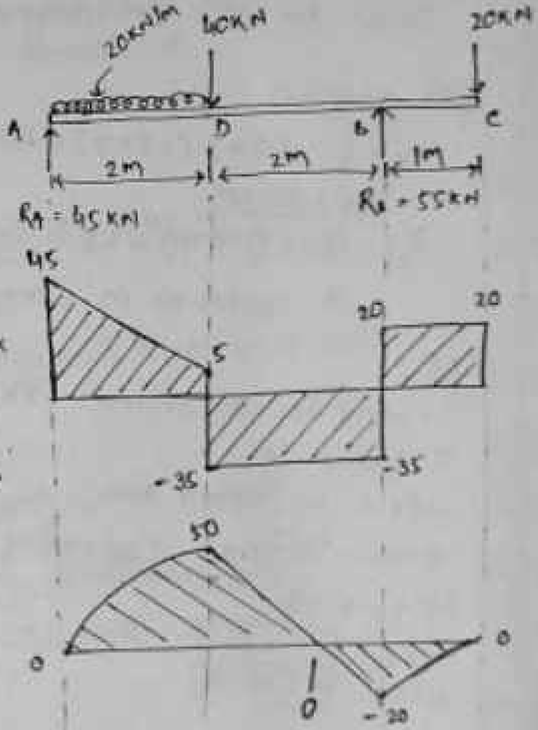
at D, $x = 1.5\,m$

$SF_D = \boxed{-55\,kN}$ $BM_D = 55\times1.5 = \boxed{82.5\,kN\text{-}m}$

problem:-

Draw the SF & BM diagram for the overhanging beam as shown in fig. Clearly indicate point of contraflexure.

Soln



$R_A + R_B = 20 + 40 + 20\times2 = 100\,kN$

$\Sigma M_A = 0$

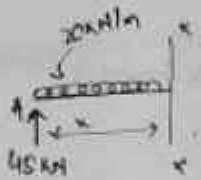
$20\times5 - R_B\times4 + 40\times2 + 20\times2\times1 = 0$

$R_B = 220/4 = \underline{55\,kN}$

$R_A = \underline{45\,kN}$

AD

Consider a cross section xx at a distance ~~from~~ x from A.



$SF_x = 45\,kN - (20\times x)\,kN$

$BM_x = 45\times x - 20\times x\times x/2$

at A, $x = 0$

$SF_A = \boxed{45\,kN}$

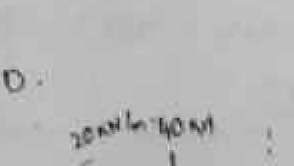
$BM_A = 45\times0 - 20\times0 = \boxed{0\,kN\text{-}m}$

at D, $x = 2m$

$SF_D = 45 - 20\times2 = \boxed{5\,kN}$

$BM_D = 45\times2 - 20\times2\times2/2 = 90 - 40 = \boxed{50\,kN\text{-}m}$

BD Consider a cross section xx at a distance x from D.



$SF_x = 45 - (20\times2) - 40 = -35\,kN$

$BM_x = 45(2+x) - [20\times2\times(1+x)] - 40\times x\ \ kN\text{-}m$

at D, $x = 0$

$SF_D = \boxed{-35\,kN}$

$BM_D = 45\times2 - 20\times2\times1 = 50\,kN\text{-}m$

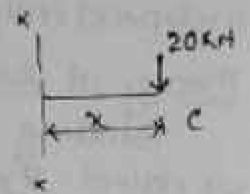
at B, x = 2m

$SF_B = \boxed{-35\,kN}$

$BM_B = 45(2+2) - [20\times2\times(1+2)] - 40\times2 = \boxed{-20\,kN\text{-}m}$

BC

Consider a section xx at a distance 'x' from C.



$SF_x = 20\,kN$

$BM_x = -20\times x\ kN\text{-}m$

at C, x = 0

$SF_C = \boxed{20\,kN}$, $BM_C = \boxed{0\,kN\text{-}m}$

at B, x = 1m

$SF_B = \boxed{20\,kN}$; $BM_B = -20\times1 = \boxed{-20\,kN\text{-}m}$

Point of contraflexure:-

It lies between BD cross section.

Moment eqn at this cross section: [As at pt of contraflexure moment = 0]

$45(2+x) - [20\times2\,(1+x)] - 40\times x = 0$

$\Rightarrow 90 + 45x - 40 - 40x - 40x = 0$

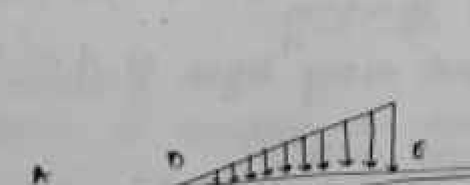
$\Rightarrow 50 - 35x = 0 \Rightarrow x = 50/35 = \boxed{1.42\,m}$

* point of contraflexure lies 1.42m from pt D & 3.42 m from pt A

Exercise problem:-

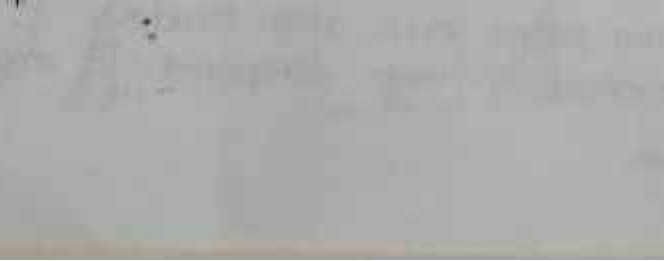
→ Draw the SF & BM diagram for following figures. Also indicate point of contraflexure if exists.

(a)



(b)

(c)

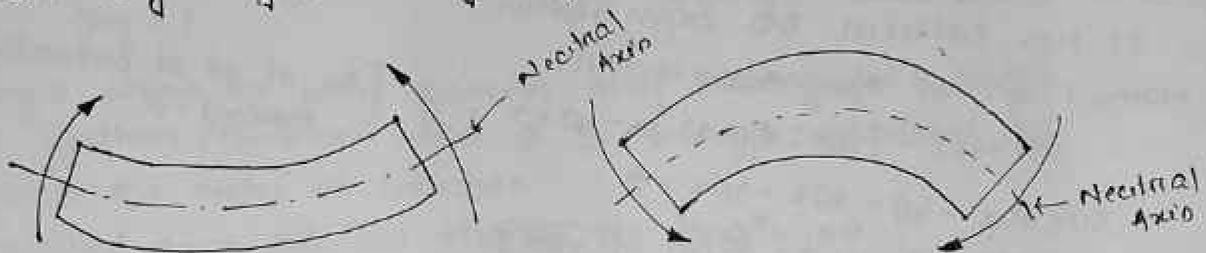


Introduction:-

An seen in the chapter shear force & bending moment we had seen that the beams are subjected to shear forces and bending moments. In order to resist these the beams develope ~~bending~~ stresses in it. Here we are interested to find out those stresses. The stresses due to bending moment and shear forces are found independently.

Theory of Simple bending:-

Bending is usually associated with shear but for simplicity we neglect the effect of shear stress and consider only moment to find the bending stresses. Such a theory which deals with finding stresses at a section due to pure moment is called simple bending moment theory. Due to bending the beam either sags (fig 1) or hogs (fig 2)



(Sagging Moment) (Hogging Moment)

In case of sagging moment fibres at bottom get stretched and hence at lower elements are subjected to tension. and the upper side fibres get compressed and hence an element in upper portion is under compression. Vice versa holds good for hogging moment. So Due to bending moment tensile stress developes in one portion and compression stress developes in other portion across the depth. In between these two portions, there is a layer where the stress nature changes from tension to compression or at that layer zero is called neutral layer. The trace of that layer is called neutral axis.

Assumptions in Simple theory of bending:-

→ The beam is initially straing and every layer of it is free to expand or contract

→ The material is homogeneous and isotropic.

→ young's Modulus is same in tension and compression.

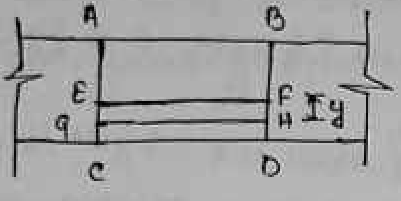
→ Stresses are within elastic limit

→ plane section remains plane even after bending.

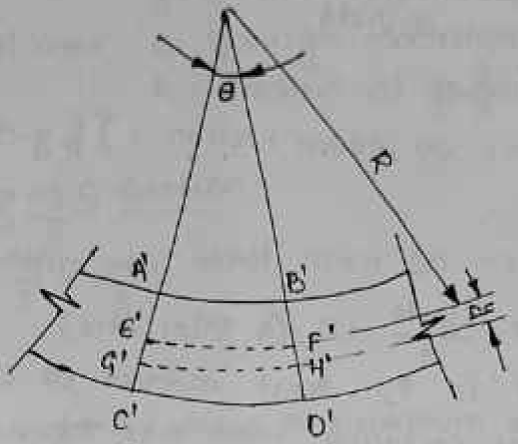
→ The radius of curvature is large compared to depth of beam.

Relationship between bending stresses and Radius of curvature:-

consider a portion of beam between sections AC and BD as shown in figs. Let EF be the neutral axis and GH an element at a distance y from neutral axis.



(Before Bending)
(fig-3)



(After Bending)
(fig-4)

In fig 4 shown the same portion after bending. Let R be the radius of gyration (curvature) and ϕ be the angle subtended by C'A' & B'D' at the centre of curvature.

The neutral axis does not change its length so

$$EF = E'F' = R\phi$$

Now, strain in layer GH = $\dfrac{\text{Final length} - \text{Original length}}{\text{Original length}}$

$$\boxed{\cancel{\frac{E'F' - EF}{EF}}} = \frac{G'H' - GH}{GH}$$

But $EF = GH = R\phi$, $GH' = (R+y)\phi$

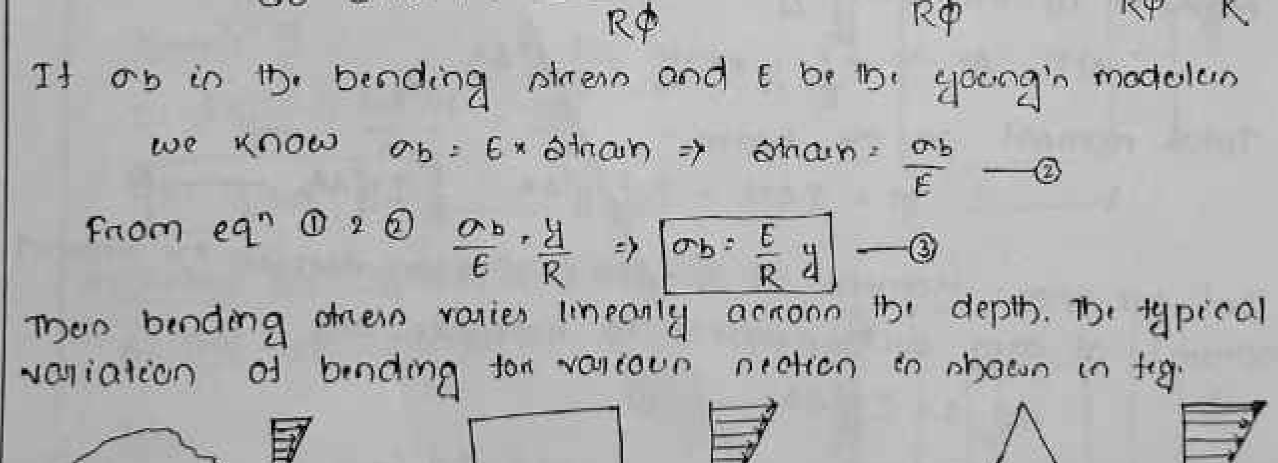
$$\text{so strain} = \frac{(R+y)\phi - R\phi}{R\phi} = \frac{\cancel{R\phi} + y\phi - \cancel{R\phi}}{R\phi} = \frac{y\phi}{R\phi} = \frac{y}{R} \quad \text{--- (1)}$$

If σ_b is the bending stress and E be the young's modulus

we know $\sigma_b = E \times \text{strain} \Rightarrow \text{strain} = \dfrac{\sigma_b}{E}$ --- (2)

From eq^n (1) & (2) $\dfrac{\sigma_b}{E} = \dfrac{y}{R} \Rightarrow \boxed{\sigma_b = \dfrac{E}{R}y}$ --- (3)

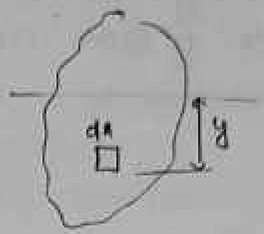
Thus bending stress varies linearly across the depth. The typical variation of bending for various section is shown in fig.



Note:-

The neutral axis coincides with the ~~centroid~~ centroid of the cross-section.

proof: Let σ be the ~~force~~ stress on dA



Force $= \sigma \times dA$

Total force $= \Sigma \sigma dA$

But $\sigma = \frac{E}{R} y$

Total force on cross section $= \Sigma \frac{E}{R} y \cdot dA$

$= \frac{E}{R} \Sigma y \cdot dA$

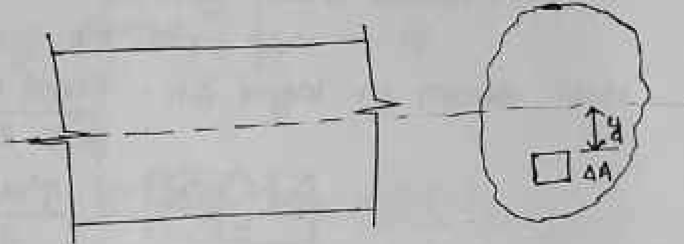
$\because$ there is no axial force on member so $\Sigma \frac{E}{R} y \cdot dA = 0 \Rightarrow \Sigma y \cdot dA = 0$

i.e $\frac{\Sigma y \cdot dA}{A} = 0$ (A = total Area)

$\because \Sigma y dA$ is the first moment of area about N.A. $\Sigma y dA / A$ is the distance of centroid from NA then Neutral axis coincide with centroid of crosssection.

Relationship between Moment and Radius of curvature:-

Consider an elemental area ΔA at distance y from neutral axis in the beam which is shown in fig.



As we know σ_b on this element is given by

$\sigma_b = \frac{E}{R} y$ —— (from eqn ③)

$\therefore$ Force on this element

$\Delta F = \sigma_b \times \Delta A$

$= \frac{E}{R} \cdot y \cdot \Delta A$

Moment of the resisting force about neutral axis

$$\Delta M = \Delta F \cdot y = \frac{E}{R} \cdot y \cdot \Delta A \cdot y = \frac{E}{R} y^2 \Delta A$$

Total moment on the beam

$$M = \Sigma \Delta M = \Sigma \frac{E}{R} y^2 \Delta A = \frac{E}{R} \Sigma y^2 \Delta A \quad \text{—— ④}$$

But we know Moment of Inertia (I) of an area is the second moment of area with respect to reference axis

$$\text{so } I = \Sigma y^2 \Delta A \quad \text{—— ⑤}$$

From eqn ④ & ⑤

$$\boxed{M = \frac{E}{R} I} = \boxed{\frac{M}{I} = \frac{E}{R}} \quad \text{—— ⑥}$$

From eq^n (5) & (6)

$$\boxed{\frac{M}{I} = \frac{\sigma_b}{y} = \frac{E}{R}} \quad \text{——— (7)}$$

where,

M : Bending Moment
I = M.I about centroidal axis
σ_b : Bending stress
y = Distance from neutral axis
E : Young's modulus
R : Radius of curvature

Moment carrying capacity of a Section:-

From bending equation $\frac{M}{I} = \frac{\sigma_b}{y}$

$$\sigma_b = \frac{M}{I} y$$

This eq^n shows that bending stress is maximum at the extreme surfaces as the distance from neutral axis (y) is maximum. So during design care should be taken that the permissible stress (σ_{per}) should be less than maximum stress (σ_{max})

$$\boxed{\sigma_{max} < \sigma_{per}} \quad \text{——— (8)}$$

Now $\frac{M}{I} = \frac{\sigma_b}{y} \Rightarrow \sigma_b = \frac{M}{(I/y)} \Rightarrow \boxed{\sigma_b = \frac{M}{Z}}$ ——— (9)

Z : Section modulus $= I/y$

Section Modulus value for various standard cross section:-

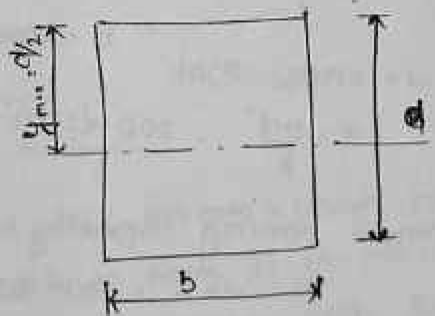
(i) Rectangular cross section:-

$$I = \frac{bd^3}{12}$$

$$y_{max} = \frac{d}{2}$$

$$Z = I/y_{max} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6}$$

$$\boxed{Z = \frac{1}{6} bd^2} \quad \text{——— (10)}$$



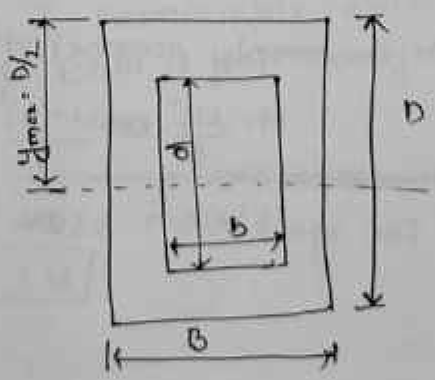
(ii) Hollow Rectangular cross section:-

$$I = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{BD^3 - bd^3}{12}$$

$$y_{max} = D/2$$

$$Z = I/y_{max} = \frac{BD^3 - bd^3}{12} \Big/ D/2$$

$$\boxed{Z = \frac{1}{6} \frac{BD^3 - bd^3}{D}}$$

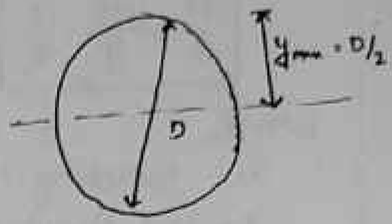


(iii) Circular cross section

$I = \frac{\pi}{64} d^4$

$y_{max} = d/2$

$Z = I/y_{max} = \frac{\pi/64 \, d^4}{d/2} = \boxed{\pi/32 \, d^3}$

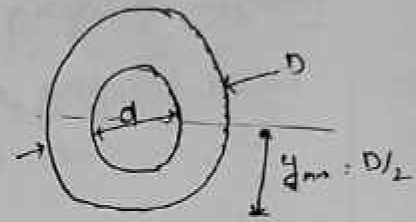


(iv) Hollow circular section:-

$I = \pi/64 \, (D^4 - d^4)$

$y_{max} = D/2$

$Z = I/y_{max} = \frac{\pi/64 \, (D^4 - d^4)}{D/2} = \boxed{\pi/32 \, (D^4 - d^4)}$

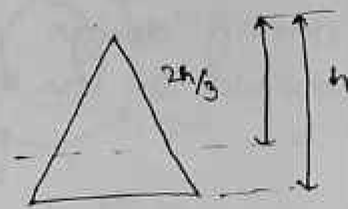


(v) Triangular cross section:-

$I = \frac{bh^3}{36}$

$y_{max} = 2h/3$

$Z = I/y_{max} = \frac{bh^3/36}{2h/3} = \boxed{\frac{bh^2}{24}}$



problem:-

A simply supported beam of span 6m has a cross section of 200mm × 250 mm. If the permissible stress is 20 N/mm², Find

(a) Maximum intensity of UDL it can carry

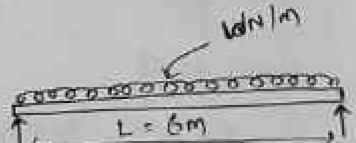
(b) Maximum concentrated load P acting at the centre

Sol^n

$b = 200mm \quad d = 250mm$

(a) we know that

$Z = \frac{bd^2}{6} = \frac{200 \times (250)^2}{6} = 2083333.33 \, mm^3$



Moment carrying capacity

$M = \sigma_b Z = 20 \times 2083333.33$

$= \boxed{41666666.67 \ N\text{-}mm}$

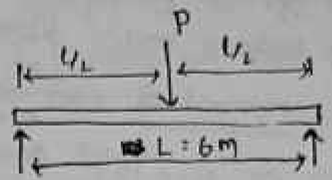
For a simply supported beam with UDL Maximum bending moment

$M = \frac{wL^2}{8} = \frac{1000 w \times (6)^2}{8} = 4500 \, w \ N\text{-}mm$

For equilibrium $4500 w = 41666666.67$

$\boxed{w = 9259.25 N}$

(b) Bending moment (M) = $\frac{Pab}{L} = \frac{P \cdot l/_2 \cdot l/_2}{l}$



$$M = \frac{Pl}{4} = \frac{P \times 6}{4} = 1.5P \ N\text{-}m$$

$$= 1500 P \ N\text{-}mm$$

Equating $1500P = 41666666.67 \ N\text{-}mm$

$$\boxed{P = 27777.77 N}$$

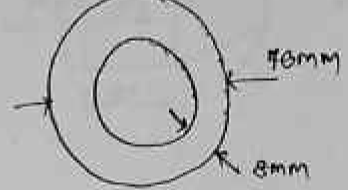
problem:-

A circular pipe of external diometer 70 mm and thickness 8mm is used as a simply supported beam over an effective length of 2.5m. Find the maximum concentrated load that can be applied at 1.5 m from left end of the span if permissible stress in tube is 150 N/mm².

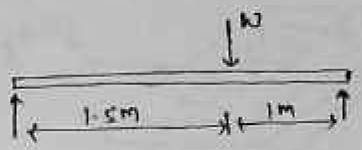
Soln

External Dia (D) = 70mm

Internal Dia (d) = 70 - 2×8 = 54mm



$$z = \frac{\frac{\pi}{32}(70^4 - 54^4)}{70} = 21748.438 \ mm^3$$



Moment carrying capacity

$$M = \sigma_b z$$
$$= 150 \times 21748.438$$
$$= 3262265.71 \ N\text{-}mm$$

Maximum moment developed at the member

$$M' = \frac{Wab}{L} = \frac{W \times 1500 \times 1000}{2500} = 600 W \ N\text{-}mm$$

For equilibrium $M' = M$

$$600W = 3262265.71$$

$$\boxed{W = 5437.10 N}$$

problem:-

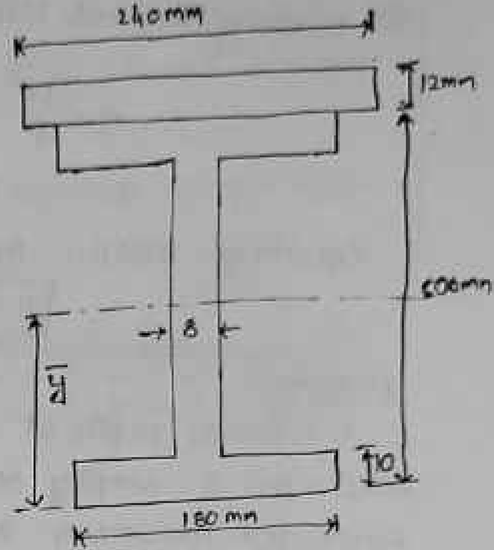
A symmetric I-section has flanges of size 180 mm × 10mm and its overall depth is 500mm. Thickness of web is 8mm. It is strengthened with a plate of size 240 mm × 12mm on compression side. Find the moment of resistance of the section, if permissible stress is 150 N/mm². How much uniformly distributed load it can carry if it is used as a cantilever of span 3m.

Let $\bar{y}$ be the distance of centroid from the bottom most fibre.

$$\bar{y} = \frac{\text{Moment of area about bottom fibre}}{\text{Total area}}$$

$$= \frac{240\times12\times506 + 180\times10\times495 + 480\times8\times250 + 180\times10\times5}{240\times12 + 180\times10 + 480\times8 + 180\times10}$$

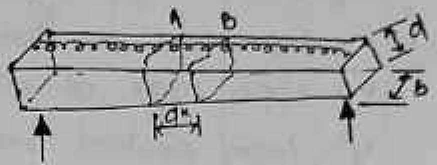
$$= \frac{3317280}{10320} = 321.442 \text{ mm}$$



Shearing Stresses In Beams:-

we had already seen that beams are subjected to BM & S.F. Now the relation between S.F & BM are $\boxed{dM = F_s\, dx}$. Now we will analyse ——① the stresses induced due to shearing forces

Consider a simply supported beam is subjected to an UDL throughout the crossection. For analysis consider a small element 'dx' of the beam.

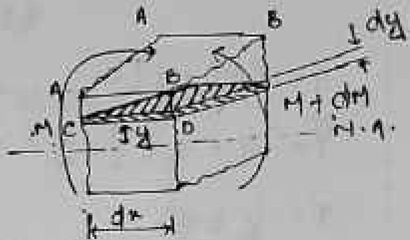


Here $AB = dx$.

Let Resisting moment at $A = M$

Resisting moment at $B = M + dM$

Consider a fiber CD having thickness 'dy' at a distance of 'y' from neutral axis.



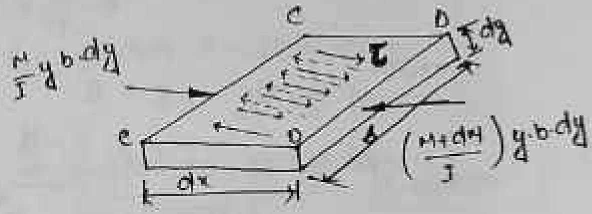
Bending stress at $C\text{-}C = \frac{M}{I}y$ ——②

Force acting at $C\text{-}C = \frac{M}{I}y \times \text{Area}$

$= \frac{M}{I}y \times b \times dy$ ——③

|ly force acting at D-D.

$\frac{M+dM}{I}\, y \cdot b \cdot dy$ ——④



Force inbalance between C-D

$\frac{M+dM}{I}\, y \cdot b \cdot dy - \frac{M}{I}\, y \cdot b \cdot dy$

$= \frac{dM}{I}\, y \cdot b \cdot dy$ ——⑤

Total force acting on upper side of the crossection from N.A.

$$F = \int_0^{d/2} \frac{dM}{I}\, y \cdot b \cdot dy = \frac{dM}{I}\int_0^{d/2} (b \cdot dy)\, y = \frac{dM}{I}\int_0^{d/2} dA \cdot y = \frac{dM}{I}(A\bar{y}) \quad ——⑥$$

A = Area of beam above neutral axis

$\bar{y}$ = Dist. between C.G of A to N.A.

If 'τ' is the intensity of shear stress then the corresponding shear force produced on the plane CCDD.

$F = \tau \times b \times dx$. ——⑦

As $F_s = \frac{dM}{dx}$

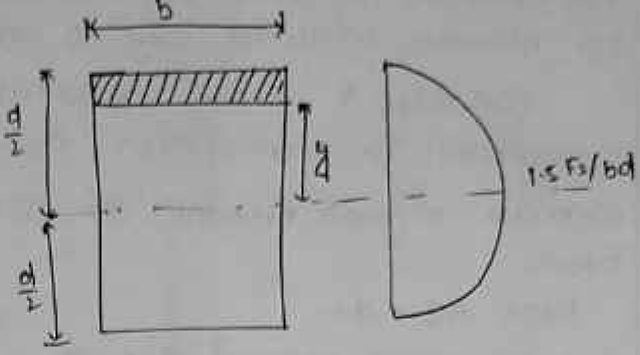
For equilibrium eqⁿ⑥ = eqⁿ⑦

$\tau \times b \times dx = \frac{dM}{I}\, A\bar{y}$

$\Rightarrow \frac{dM}{dx} = \frac{\tau b I}{A\bar{y}} \Rightarrow F_s = \frac{\tau b I}{A\bar{y}} \Rightarrow \boxed{\tau = \frac{F_s A\bar{y}}{bI}}$ ——⑧

Shear stress across a few standard cross section:-

→ Rectangular Section:-



Consider a rectangular crosssection having base 'b' depth 'd'.

Let us take an element at the upper edge at a distance 'y' from neutral axis.

we know $\tau = \frac{F_s A \bar{y}}{bI}$

$A = (d/2 - y)\, b$

$\bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = \frac{1}{2}\left(d/2 + y\right)$

$I = \frac{1}{12} b d^3$

Now $\tau = \frac{F_s \times (d/2 - y)\, b \times \frac{1}{2}(d/2 + y)}{b \cdot \frac{1}{12} b d^3}$ (parabolic)

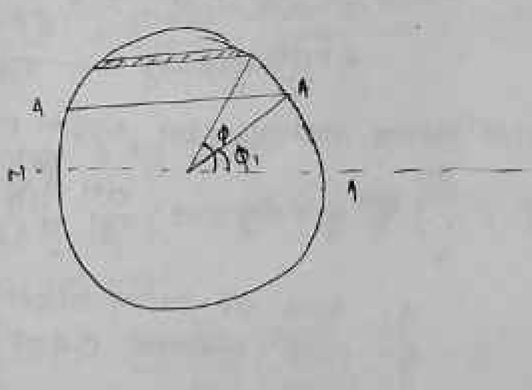
Now

at $y = d/2$, $\tau = 0$

at $y = 0$, $\tau = \frac{F_s \cdot \frac{d}{2} \cdot b \cdot \frac{1}{2} \cdot \frac{d}{2}}{\frac{b^2 d^3}{12}} = \frac{6F_s}{4bd} = \boxed{1.5\frac{F_s}{bd}} = \boxed{1.5 \times q_{avg}}$

→ Circular Section:-

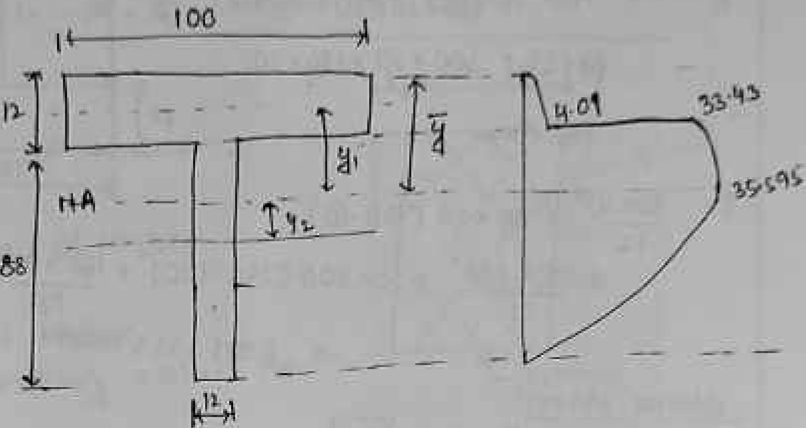
Consider a section AA for which area moment of inertia is required to found out about N·A



problem:-

A symmetrical T-section is subjected to a shear force of 30kN. Draw the shear force distribution across the depth marking values at salient points.

Sol^n



Let $\bar{y}$ be the distance of CG from the top fibre

Now

$$\bar{y} = \frac{(100\times12)\times6 + (88\times12)(44+12)}{(100\times12) + (88\times12)}$$

$$= 29.404\text{mm}$$

M.I about N.A

$$I = \frac{100\times12^3}{12} + 100\times12\times(29.404-6)^2 + \frac{12\times(88)^3}{12} + (44+12-29.404)^2$$

$$= 2100127.3 \text{ mm}^4$$

Shear stress at

→ Top of flange = $\boxed{0}$

→ Bottom of flange.

Area = $100\times12 = 1200$ mm^2

(y_1) Distance of CG (Flange) from N.A = $29.404-6 = 23.404$mm

· width of flange(b) = 100 mm

$$\text{Shear stress } (\tau)_{\text{Flange Bottom}} = \frac{F_s A y_1}{bI} = \frac{30\times10^3\times1200\times23.404}{100\times2100127.3} = \boxed{4.01 \text{ N/mm}^2}$$

→ Top of web =

~~Area (A) = 12 (29.404 - 12)~~

width at top of web (b) = 12mm

$$\text{Shear stress } (\tau)_{\text{top web}} = \frac{F_s A y_1}{bI} = \frac{30\times10^3\times1200\times23.404}{12\times2100127.3} = \boxed{33.43 \text{ N/mm}^2}$$

→ Neutral axis:-

Ay above N.A = Ay of flange + Ay above NA in web.

$$= 1200(29.404-6) + 12\times(29.404-12)\times\tfrac{1}{2}(29.404-12)$$

$$= 29902.195 \text{ mm}^3$$

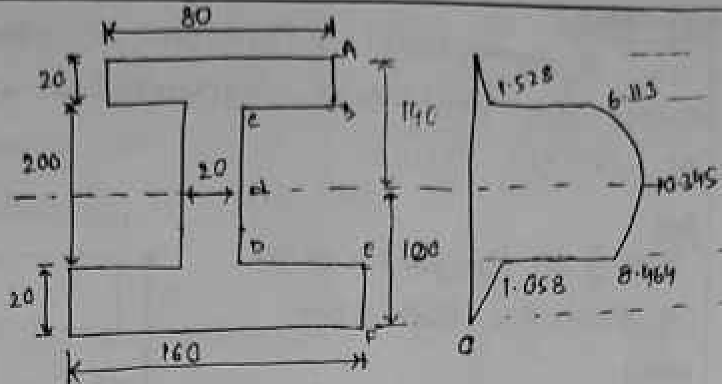
$$\text{Shear stress } (\tau_{NA}) = \frac{F_s A y}{bI} = \frac{30\times10^3\times29902.195}{12\times2100127.3} = \boxed{35.595 \text{ N/mm}^2}$$

→ Bottom of web: 0

problem:-

An unsymmetrical I-section as shown in fig subjected to a shear force of 40kN. Draw the shear stress variation diagram.

Distance of neutral axis from top fibre



$$\bar{y} = \frac{80\times20\times10 + 200\times20\,(100+20) + 160\times20\,(20+200+10)}{80\times20 + 200\times20 + 160\times20}$$

$= 140\ mm$

$$I = \frac{80\times10^3}{12} + 80\times20\,(140-10)^2 + \frac{20\times200^3}{12} + 20\times200\,(140-120)^2 + \frac{160\times20^3}{12} + 160\times20\times(230-140)^2$$

$= 68053333\ mm^4$

Shear stress:-

→ Top of top flange $= \boxed{0}$

→ Bottom of top flange =

$Ay = 80\times20\times(140-10)$, $b = 80$

$\tau = \frac{F_s}{bI}Ay = \frac{40\times1000}{80\times68053333}\times80\times20\,(140-10) = \boxed{1{\cdot}528\ N/mm^2}$

→ Top of web - $(b = 20)$

$\tau = \frac{F_s}{bI}Ay = \frac{40\times1000}{20\times68053333}\times80\times20\,(140-10) = \boxed{6{\cdot}113\ N/mm^2}$

→ Neutral Axis

Ay = Ay of top flange + Ay of portion above NA

$= (80\times20)\times(140-10) + 20\times(140-20)\times\left(\frac{140-20}{2}\right)$

$= 352000\ mm^3$

$\tau_{NA} = \frac{F_s}{bI}Ay = \frac{40\times1000}{20\times68053333}\times352000 = \boxed{10{\cdot}345\ N/mm^2}$

→ ~~Top of~~ Bottom of web $(b = 20)$

$Ay = 160\times20\times(100-10)$,

$\tau = \frac{F_s}{bI}Ay = \frac{40\times1000}{20\times68053333}\times160\times20\,(100-10) = \boxed{8{\cdot}464\ N/mm^2}$

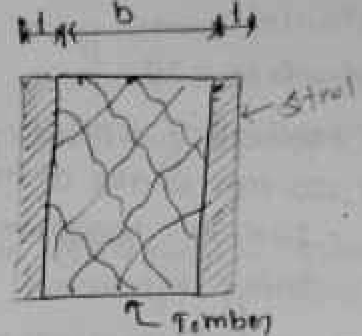
→ Top of Bottom flange, $(b = 160)$

$\tau = \frac{F_s}{bI}Ay = \frac{40\times1000}{160\times68053333}\times160\times20\,(100-10) = \boxed{1{\cdot}058\ N/mm^2}$

→ Bottom of Bottom flange $= \boxed{0}$

Composite Beams:

Till now we had analysed about the beams of same material but in practice there are beams made up of more than one matl. is also found. Such beams are called <u>composite beams</u> or <u>flitched</u> beams. Very common example of this type of beams are timber beams with steel plates.



Let us consider

E_1 = Modulus of elasticity of timber

I_1 = M.I. of timber

M_1 = Moment of resistance of timber

σ_1 = Stress in timber

Z_1 = Section modulus of timber

$E_2, I_2, M_2, \sigma_2, Z_2$ = corresponding values for steel.

We know that the moment of resistance

$$M_1 = \sigma_1 \times Z_1 \quad , M_2 = \sigma_2 \times Z_2 \qquad \text{——— (1)}$$

Total moment of resistance for the composite beam

$$M = M_1 + M_2 = (\sigma_1 \times Z_1) + (\sigma_2 \times Z_2) \qquad \text{——— (2)}$$

We also know that at any distance from neutral axis, the strain in both materials will be same

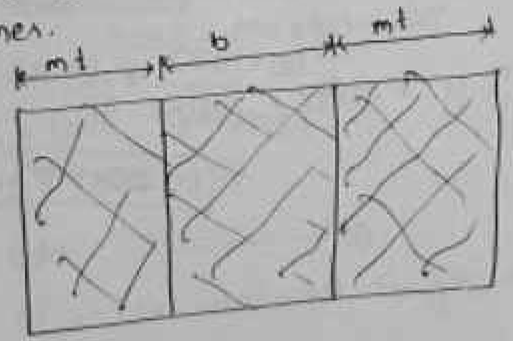
$$\frac{\sigma_1}{E_1} = \frac{\sigma_2}{E_2} \Rightarrow \sigma_1 = \sigma_2 \frac{E_1}{E_2} = m \times \sigma_2 \qquad \text{——— (3)}$$

$m = E_1/E_2$ = Modular ratio.

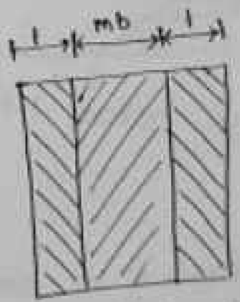
Thus moment of resistance of element of material-1 is E_1/E_2 times that of element of material-2 placed at same distance from neutral axis. We can replace the material 1 by material-2

$$M_1 = \sigma_1 Z_1 = m \cdot \sigma_2 Z_1 \quad [\text{eq}^n (3)]$$

So the width of section can be replaced by material-2 by m times.



Material-1 = Steel
Material-2 = Wood

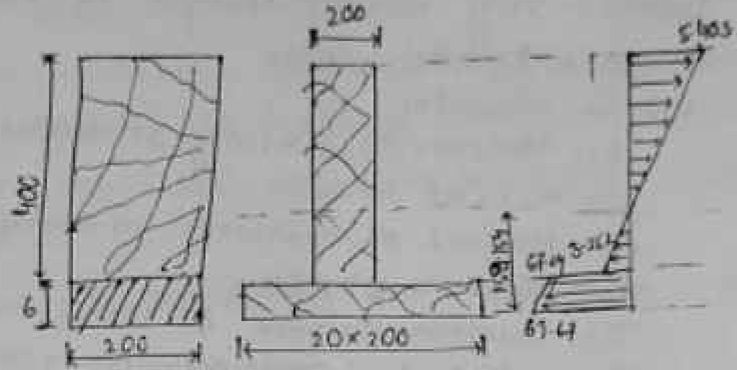


Material-1 = wood
Material-2 = Steel

Here we are converting one material by another. That means we are bringing an equivalent value of σ_1 in terms of σ_2.

problem:

A 200 mm wide and 400 mm deep timber beam is strengthened with 6 mm thick and 200 mm wide steel plate as shown. Determine the extreme fibre stresses, if the section is subjected to a moment of 40 kN·m take $E_s/E_t = 20$



Sol^n

Since $E_s/E_t = 20$

So equivalent timber section for 200 mm width of timber

$= 20 \times 200 = 40000$ mm

Considering equivalent section

$$\bar{y} = \frac{200 \times 400\,(200+6) + 20 \times 200 \times 6 \times 3}{200 \times 400 + 200 \times 20 \times 6}$$

$$= \frac{16552000}{104000} = 159.154\text{mm}$$

$$I = \frac{200 \times 400^3}{12} + 200 \times 400\,(206 - 159.154)^2 + \frac{20 \times 200 \times 6^3}{12} + 20 \times 200 \times 6\,(159.154 - 3)^2$$

$$= 1.82752 \times 10^9 \text{ mm}^4$$

Distance of top fibre from N-A $= 406 - 159.154 = 246.846$ mm

Distance of junction from N-A $= 159.154 - 6 = 153.154$mm

Bending stress:-

at top fibre $= \frac{M}{I}y = \frac{40 \times 1000}{1.82752 \times 10^9} \times 246.846 = \boxed{5.403\text{ N/mm}^2}$

at the junction (on wood) $= \frac{M}{I}y = \frac{40 \times 1000}{1.82752 \times 10^9} \times 153.154$
$= \boxed{3.352\text{ N/mm}^2}$

At the junction (on steel) $= \frac{E_s}{E_t}\frac{M}{I}y = 20 \times 3.352$
$= \boxed{67.04\text{ N/mm}^2}$

At the bottom fibre $= \frac{E_s}{E_t}\frac{M}{I}y = 20 \times \frac{40 \times 1000}{1.82752 \times 10^7} \times 159.154$
$= \boxed{69.670\text{ N/mm}^2}$

Introduction:-

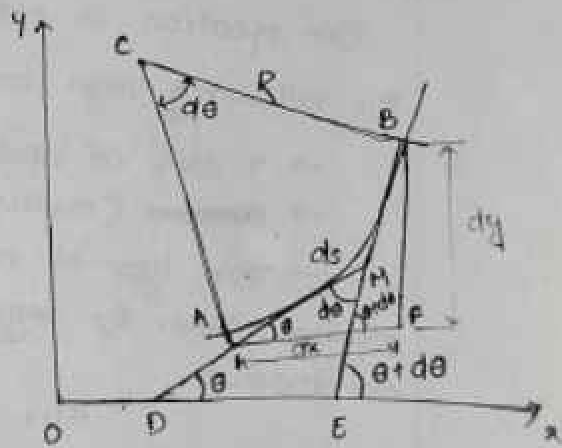
Materials used in beams are elastic in nature hence under the action of loads the beam axis deflects. A designer has to decide about beam dimension not only based on strength requirement but also from the consideration of deflection. which should be with in prescribed limit.

In mechanical components excessive deflection may cause misallignment and non performance of machine. In building excessive deformation gives rise to psychological unrest and some times breaking of flooring, ceiling or roofing materials.

There are various methods of calculating beam deflections. them are
- → Double integration / Direct integration / Macaulay's method.
- → conjugate beam method
- → Castigliano's theorem.

Differential Equation for Deflection:-

Consider an elemental length AB = ds as shown in fig. Let tangents drawn at A & B make angles θ and θ+dθ with x-axis and intersect it at D and E. Let M be the intersection point of these two tangents.



$\angle EDM = \theta$, $\angle xEM = \theta + d\theta$

Then,

$\angle xEM = \angle DME + \cancel{\theta} = \cancel{\theta} + d\theta$

$\Rightarrow \angle DME = d\theta$

Also,

$\angle DME + \angle AMB = 180^\circ$

$\angle AMB + \angle ACB = 360^\circ - (\angle CAM + \angle CBM)$

$= 360^\circ - (90+90) = 180^\circ$

$\Rightarrow \angle DME + \cancel{\angle AMB} = \cancel{\angle AMB} + \angle ACB$

$\Rightarrow \angle DME = \angle ACB = d\theta$

$ds = R\,d\theta$ ——— (1)

Since ds is an elemental length, treating ABF as a triangle,

$\frac{ds}{dx} = \sec\theta$ ——— (2)

$\frac{dy}{dx} = \tan\theta$ ——— (3)

$\frac{1}{R} = \frac{d\theta}{ds}$ ——— (4)

Differentiating eqⁿ ③ w.r.t 'x' we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta) = \sec^2\theta \, \frac{d\theta}{dx} = \sec^2\theta \, \frac{d\theta}{ds}\,\frac{ds}{dx}$$

$$= \sec^2\theta \times \frac{1}{R} \times \sec\theta = \sec^3\theta \times \frac{1}{R}$$

$$\frac{1}{R} = \frac{d^2y/dx^2}{\sec^3\theta} = \frac{d^2y/dx^2}{(\sec^2\theta)^{3/2}} = \frac{d^2y/dx^2}{(1+\tan^2\theta)^{3/2}} \quad [\sec^2\theta = 1+\tan^2\theta]$$

$$= \frac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \quad \text{———— ⑤}$$

In beams, deflections are small and hence (dy/dx) is also small. Therefore we can neglect $\left(\frac{dy}{dx}\right)^2$

Hence $\frac{1}{R} = \frac{d^2y}{dx^2}$ ———————— ⑥

we know that $\frac{M}{I} = \frac{E}{R} \Rightarrow \frac{1}{R} = \frac{M}{EI}$ ———— ⑦

$\Rightarrow \frac{M}{EI} = \frac{d^2y}{dx^2} \Rightarrow M = EI\left(\frac{d^2y}{dx^2}\right)$ ———— ⑧

This equation is called differential equation of deflection.

The following sign conventions are used in deriving eqⁿ ⑧

→ y-axis is upward.

→ ~~Concave~~ Curvature is concave towards +ve y-axis.

→ This type of curvature occurs in the beam due to sagging moment. Hence the sagging moment is to be considered as the +ve moment

Here EI = flexural Rigidity.

Double Integration Method:-

In this method, moment M, at any distance x from one of the supports is written with the sagging moment as positive. Then from equation

$$EI\frac{d^2y}{dx^2} = M$$

$$EI\frac{dy}{dx} = \int_0^x M\,dx + C_1$$

$$EIy = \int_0^x\int_0^x M\,dx + C_1x + C_2$$

The constants C_1 and C_2 are found by making use of boundary conditions

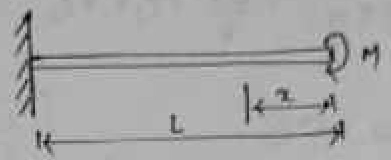
(a) At simply supported roller ends, deflection $y=0$

(b) At fixed ends, deflection $y=0$, $dy/dx=0$

(c) At point of symmetry $dy/dx=0$

Few different cases:-

(i) Cantilever subjected to moment at free end:-



A cantilever beam of length 'L', flexural rigidity EI, subjected to hogging moment M. Taking moment at a distance x

$M_x = -M$

$EI\frac{d^2y}{dx^2} = -M$ —① $\Rightarrow EI\frac{dy}{dx} = -Mx + C_1$ —② & $EIy = -\frac{Mx^2}{2} + C_1x + C_2$ —③

at $x = L$, deflection $y=0 \Rightarrow \frac{dy}{dx}=0$

Now eqⁿ ②

$EI\frac{dy}{dx} = -Mx + C_1 \Rightarrow 0 = -Mx + C_1 \Rightarrow -ML + C_1 = 0 \Rightarrow \boxed{C_1 = ML}$

eqⁿ ③

$EIy = -\frac{Mx^2}{2} + C_1x + C_2 \Rightarrow 0 = -\frac{ML^2}{2} + ML^2 + C_2 \Rightarrow C_2 = \frac{ML^2}{2} - ML^2 = \boxed{\frac{-ML^2}{2}}$

Now $EI\frac{dy}{dx} = -Mx + ML = M(L-x)$

$EIy = -\frac{Mx^2}{2} + MLx - \frac{ML^2}{2} \Rightarrow M\left[-\frac{x^2}{2} + Lx - \frac{L^2}{2}\right]$

at free end $x=0$

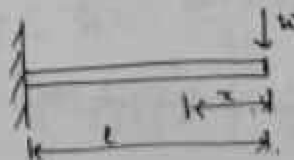
$EI\frac{dy}{dx} = -Mx + C_1 = C_1$

$\Rightarrow \frac{dy}{dx} = \frac{C_1}{EI} = \frac{ML}{EI}$

$\cancel{y = \int \frac{ML}{EI}dx = \frac{M}{EI}\left(\frac{L^2}{2}\right)} \Rightarrow y = \frac{M}{EI}\left[-\frac{x^2}{2} + Lx - \frac{L^2}{2}\right]$

$\Rightarrow \boxed{y = \frac{-ML^2}{2EI}}$

(ii) Cantilever subjected to pt. load at free end:-



From fig

$M_x = -Wx$

$EI\frac{d^2y}{dx^2} = -Wx$ —①

$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1$ —②

$EIy = -\frac{wx^3}{6} + C_1 x + C_2$ —— (3)

at $x = L$, $y = 0 \Rightarrow \frac{dy}{dx} = 0$

eq^n (2)

$EI\frac{dy}{dx} = -\frac{wx^2}{2} + C_1 \Rightarrow 0 = -\frac{wL^2}{2} + C_1 \Rightarrow \boxed{C_1 = \frac{wL^2}{2}}$

Now $EI\frac{dy}{dx} = -\frac{wx^2}{2} + \frac{wL^2}{2}$

at eq^n (3)

$EIy = -\frac{wx^3}{6} + C_1 x + C_2 \Rightarrow 0 = -\frac{wL^3}{6} + \left(\frac{wL^2}{2}\right)L + C_2 \Rightarrow C_2 = \frac{wL^3}{6} - \frac{wL^3}{2}$

$= \frac{wL^3 - 3wL^3}{6} = \boxed{-\frac{wL^3}{3}}$

Now eq^n (3)

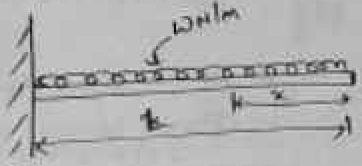
$EIy = -\frac{wx^3}{6} + \frac{wL^2}{2}x - \frac{wL^3}{3}$

at $x = 0$, $\frac{dy}{dx} = \frac{1}{EI}C_1 = \frac{1}{EI}\left(\frac{wL^2}{2}\right) = \boxed{\frac{wL^2}{2EI}}$ (Slope)

$y = \frac{1}{EI}C_2 = \frac{1}{EI}\left(\frac{-wL^3}{3}\right) = \boxed{\frac{-wL^3}{3EI}}$ (Deflection)

$\left(\frac{wL^3}{3EI} \text{ downward}\right)$

<u>Cantilever subjected to UDL</u>:-



from fig.

$M = w \times x \times \frac{x}{2} = -\frac{wx^2}{2}$

$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$ —(1), $EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$ —(2), $EIy = -\frac{wx^4}{24} + C_1 x + C_2$ —(3)

at $x = L$, $y = 0$, $\frac{dy}{dx} = 0$

eq^n (2) $EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1 \Rightarrow 0 = -\frac{wL^3}{6} + C_1 \Rightarrow \boxed{C_1 = \frac{wL^3}{6}}$

Now $EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wL^3}{6}$

eq^n (3) $EIy = -\frac{wx^4}{24} + C_1 x + C_2 \Rightarrow 0 = -\frac{wL^4}{24} + \frac{wL^3}{6}L + C_2 \Rightarrow 0 = \frac{-wL^4 + 4wL^4}{24} + C_2$

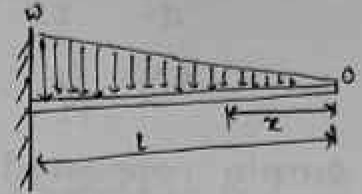
$\Rightarrow C_2 = -\frac{3wL^4}{24} = \boxed{-\frac{wL^4}{8}}$

Now $EIy = -\frac{wx^4}{24} + \frac{wL^3}{6}x - \frac{wL^4}{8}$

at $x = 0$, $\frac{dy}{dx} = \frac{1}{EI}\left(\frac{wL^3}{6}\right) = \boxed{\frac{wL^3}{6EI}}$ (Slope)

$y = \frac{1}{EI} \cdot \frac{-wL^4}{8} = \boxed{\frac{-wL^4}{8EI}}$ (Deflection)

Cantilever subjected to a UVL varying from 0 to w :-



from fig.

$$M_x = \text{load} \times \frac{x}{3} = -\frac{1}{2} \times x \times \frac{w}{L} x \times \frac{x}{3} = -\frac{wx^3}{6L}$$

$$EI \frac{d^2y}{dx^2} = -\frac{wx^3}{6L} \quad \text{——— (1)}$$

$$EI \frac{dy}{dx} = -\frac{w}{6L} \frac{x^4}{4} = -\frac{wx^4}{24L} + C_1 \quad \text{——— (2)}$$

$$EIy = -\frac{w}{24L} \frac{x^5}{5} + C_1 x + C_2 = -\frac{wx^5}{120L} + C_1 x + C_2 \quad \text{——— (3)}$$

at $x = L$, $y = 0$, $\frac{dy}{dx} = 0$

eqn (1)

$$EI \frac{dy}{dx} = -\frac{wx^4}{24L} + C_1 \Rightarrow 0 = -\frac{wL^4}{24L} + C_1 \Rightarrow \boxed{C_1 = \frac{wL^3}{24}}$$

eqn (3)

$$EIy = -\frac{wL^5}{120L} + \frac{wL^3}{24} L + C_2 \Rightarrow 0 = -\frac{wL^4}{120} + \frac{wL^4}{24} + C_2 \Rightarrow 0 = \frac{-wL^4 + 5wL^4}{120} + C_2$$

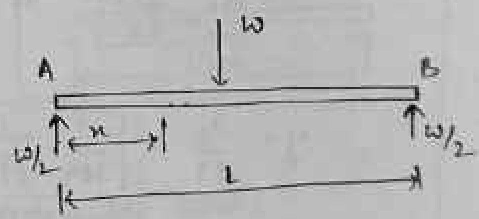
$$\Rightarrow \frac{4wL^4}{120} + C_2 = 0 \Rightarrow \boxed{C_2 = -\frac{wL^4}{30}}$$

Now $EIy = -\frac{wx^5}{120L} + \frac{wL^3}{24} x - \frac{wL^4}{30}$

at $x = 0$, $\frac{dy}{dx} = \frac{C_1}{EI} = \boxed{\frac{wL^3}{24EI}}$ (slope)

$y = \frac{C_2}{EI} = \boxed{-\frac{wL^4}{30EI}}$ (Deflection)

Simply supported beam subjected to point load at centre :-



from fig, $R_A = w/2$

$M_x = R_A x = \frac{wx}{2}$

$$EI \frac{d^2y}{dx^2} = \frac{wx}{2} \quad \text{——— (1)}$$

$$EI \frac{dy}{dx} = \frac{wx^2}{4} + C_1 \quad \text{——— (2)}$$

$$EIy = \frac{wx^3}{12} + C_1 x + C_2 \quad \text{——— (3)}$$

considering at $x = L/2$, $\frac{dy}{dx} = 0$, $y = 0$

Now eqn (2) $EI \frac{dy}{dx} = \frac{wx^2}{4} + C_1 \Rightarrow 0 = \frac{wL^2}{16} + C_1 \Rightarrow \boxed{C_1 = -\frac{wL^2}{16}}$

eqn (3) $EIy = \frac{wx^3}{12} - \frac{wL^2}{16} x + C_2 \Rightarrow 0 = \frac{wL^3}{96} - \frac{wL^3}{32} + C_2 \Rightarrow 0 = \frac{wL^3 - 3wL^3}{96} + C_2$

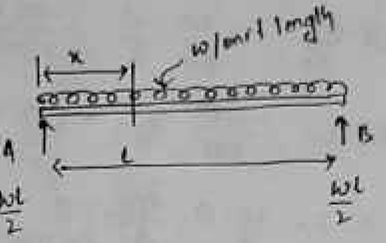
$$\Rightarrow C_2 = \frac{-2wL^3}{96} = \boxed{-\frac{wL^3}{48}}$$

Now $EIy = \frac{wx^3}{12} - \frac{wL^2}{16} x - \frac{wL^3}{48}$

at $x=0$, $\frac{dy}{dx} = \frac{C_1}{EI} = \frac{-wl^3}{16EI}$ (slope)

$y = \frac{C_2}{EI} = \boxed{\frac{-wl^3}{48EI}}$ (deflection)

Simply supported beam subjected to UDL:-



$R_A = R_B = wl/2$

$M_x = \frac{wl}{2}x - \frac{wx^2}{2}$

$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}$ — ①

$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1$ — ②

$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} + C_1x + C_2$ — ③

at $x = l/2$, $y=0$, $dy/dx = 0$

eq② $EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1 \Rightarrow 0 = \frac{wl l^2}{16} - \frac{wl^3}{48} + C_1 \Rightarrow C_1 = \frac{wl^3}{48} - \frac{wl^3}{16}$

$\Rightarrow C_1 = \frac{wl^3 - 3wl^3}{48} = \boxed{\frac{-wl^3}{24}}$

eq③

$EIy = \frac{wl l^3}{96} - \frac{wl^4}{384} - \frac{wl^3}{24}\left(\frac{l}{2}\right) + C_2$

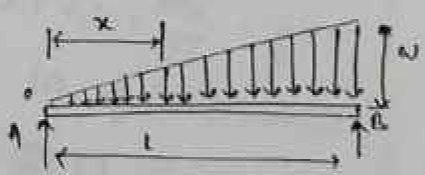
$\Rightarrow 0 = \frac{wl^4}{96} - \frac{wl^4}{384} - \frac{wl^4}{48} + C_2 \Rightarrow 0 = \frac{4wl^4 - wl^4 - 8wl^4}{384} + C_2$

$\Rightarrow \boxed{C_2 = -5/384\ wl^4}$

$y = \frac{C_2}{EI} = \boxed{\frac{-5}{384}\frac{wl^4}{EI}}$

$\frac{dy}{dx} = \frac{C_1}{EI} = \boxed{\frac{wl^3}{24EI}}$

Simply supported beam subjected to varying load ~~for~~ zero to w.



Taking moment about B

$R_A \times l = \frac{1}{2} w \times l \times \frac{l}{3} \Rightarrow R_A = \frac{wl}{6}$

$R_B = \frac{wl}{2} - \frac{wl}{6} = \frac{3wl - wl}{6} = \frac{wl}{3}$

$M_x = R_A x - \frac{1}{2}\cdot x \cdot \frac{w}{L}x \cdot \frac{x}{3} \Rightarrow \frac{wl}{6}x - \frac{wx^3}{6L}$

$EI\frac{d^2y}{dx^2} = \frac{wl}{6}x - \frac{wx^3}{6L}$ —— ①

$EI\frac{dy}{dx} = \frac{wlx^2}{12} - \frac{wx^4}{24L} + C_1$ —— ②

$$EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} + c_1x + c_2 \quad \text{——} ③$$

Assuming deflection at reaction in zero

at $x=0,\ y=0,\ \frac{dy}{dx}=0 \Rightarrow c_2=0$

$x=L,\ y=0,\ \frac{dy}{dx}=0$

eq^n-2

$\cancel{EI\frac{dy}{dx} = \frac{wLx^2}{12} - \frac{wx^4}{24L} + c_1 \Rightarrow 0 = \frac{wL^3}{12} - \frac{wL^4}{24L} + c_1} \Rightarrow \frac{wL^3}{24} - \frac{wL^3}{12} = c$

$\cancel{\Rightarrow c_1 = \frac{wL^3 - 2wL^3}{24}} = \boxed{\frac{-wL^3}{24}}$

$EIy = \frac{wL^4}{36} - \frac{wL^4}{120} + c_1L = 0 \Rightarrow \frac{wL^4(2.33)}{120} = 2c_1L$

$\Rightarrow \boxed{c_1 = -7/360\, wL^3}$

Now,

$$EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} - \frac{7}{360}wL^3x$$

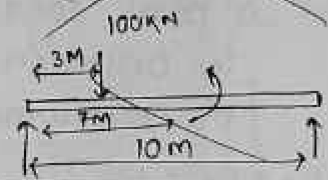
$$EI\frac{dy}{dx} = \frac{wLx^2}{12} - \frac{wx^4}{24L} - \frac{7wL^3}{360}$$

Now considering for the point where maximum deflection

$$\frac{dy}{dx} = 0$$

problem:-

~~A SSB of 10m length carries a point load of 100kN and a pure moment of 100kNm at 3m and 7m respectly from the left end. Find the slopes~~



Now
$$0 = \frac{wLx^2}{12} - \frac{wx^4}{24L} - \frac{7wL^3}{360}$$

$$0 = \frac{60wL^2x^2 - 30wx^4 - 14wL^4}{720L}$$

$\Rightarrow 60wL^3x^2 - 30wx^4 - 14wL^4 = 0$ (All divided by 30w)

$\Rightarrow x^4 - 2x^2L^3 - 7/15\, L^4 = 0$

$$x^2 = \frac{2L^2 \pm \sqrt{4L^4 - \frac{4\times7}{15}L^4}}{2}$$

$\Rightarrow L^2\left(1 \pm \sqrt{1 - 7/15}\right) = 0.2697L^2$

$\boxed{x = 0.5193L}$

Now putting value of x

$$EIy = \frac{wLx^3}{36} - \frac{wx^5}{120L} - \frac{7wL^3}{360}x$$

at $x = 0.5193L$

$$EIy = \frac{wL(0.5193L)^3}{36} - \frac{w \times (0.5193L)^5}{120L} - \frac{7wL^3}{360} \times (0.5193L)$$

$$= \frac{0.1400\,wL^4}{36} - \frac{0.0377\,L^4 w}{120} - \frac{3.635\,wL^4}{360}$$

$$EIy_{max} = -0.006523\,wL^4$$

$$\boxed{y_{max} = \frac{-0.006523\,wL^4}{EI}}$$

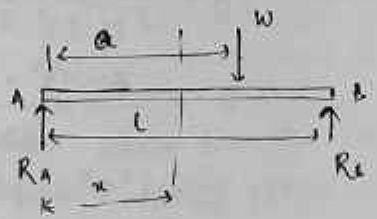
Problem:-

A beam AB of length L is simply supported at the ends and carries a point load 'w' at a distance 'a' from the left hand. Find

(i) Deflection under the load

(ii) Maximum deflection

Sol^n



$R_A + R_B = w$

$R_A \times L - w(L-a) = 0$

$\Rightarrow R_A = \frac{w(L-a)}{L}$

$R_B = w - \frac{w(L-a)}{L} = \frac{wL - wL + aw}{L} = \frac{wa}{L}$

At any point x B.M (M) $= -R_A \times x = -\frac{w(L-a)}{L}x$

$$EI\frac{d^2y}{dx^2} = \frac{-w(L-a)}{L}x \quad \text{--- (1)}$$

$$EI\frac{dy}{dx} = -\frac{w(L-a)}{L}\frac{x^2}{2} + C_1 \quad \text{--- (2)}$$

$$EIy = -\frac{w(L-a)}{L} \times \frac{x^3}{6} + C_1 x + C_2 \quad \text{--- (3)}$$

at $x = 0$, $y = 0$ $\Rightarrow C_2 = 0$ also at $x = L$, $y = 0$

$\hookrightarrow$ cond^n (1) $\qquad \hookrightarrow$ cond^n (2)

Now eq^n (3)

$$EIy = \frac{-w(L-a)}{L} \times \frac{L^3}{6} + C_1 L = 0 \quad [\text{As } y = 0] \quad [\text{putting cond}^n \text{ (2)}]$$

$\Rightarrow \dfrac{-wL^2(L-a)}{6} + C_1 L = 0 \Rightarrow C_1 = \dfrac{wL(L-a)}{6}$

Now

$$EIy = -\frac{w(L-a)}{L}\frac{x^3}{6} + \frac{wL(L-a)}{6}x \quad \text{——④}$$

$$EI\frac{dy}{dx} = -\frac{w(L-a)}{L}\frac{x^2}{2} + \frac{wL(L-a)}{6} \quad \text{——⑤}$$

(i) Maximum deflection:-

For maximum deflection $dy/dx = 0$ (slope = 0)

Now eq^n ⑤ $\quad 0 = \dfrac{-w(L-a)}{2L}x^2 + \dfrac{wL(L-a)}{6} \Rightarrow x^2 = \dfrac{wL(L-a)\times 2L}{6\times w(L-a)} = \dfrac{2L^2}{6} = \dfrac{L^2}{3}$

$\Rightarrow \boxed{x = L/\sqrt{3}}$

Now $y_{max} = \dfrac{1}{EI}\left[\dfrac{-w(L-a)}{L}\dfrac{(L/\sqrt{3})^3}{6} - \dfrac{wL(L-a)}{6}\dfrac{L}{\sqrt{3}}\right]$

$= \dfrac{1}{EI}\left[\dfrac{-w(L-a)L^2}{18\sqrt{3}} - \dfrac{wL^2(L-a)}{6\sqrt{3}}\right] = \dfrac{1}{EI}\left[\dfrac{-wL^2(L-a) - 3wL^2(L-a)}{18\sqrt{3}}\right]$

$= \boxed{\dfrac{-4wL^2(L-a)}{18\sqrt{3}\,EI}}$

(ii) Deflection under the load:-

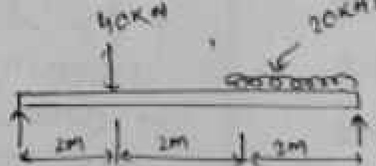
at $x = a$

$y = \dfrac{1}{EI}\left[\dfrac{-w(L-a)}{L}\dfrac{x^3}{6} + \dfrac{wL(L-a)}{6}x\right] = \dfrac{1}{EI}\left[-\dfrac{w(L-a)a^3}{6L} + \dfrac{wL(L-a)}{6}a\right]$

$= \dfrac{1}{EI}\left[\dfrac{w(L-a)a}{6}\left[L - \dfrac{a^2}{L}\right]\right] = \boxed{\dfrac{1}{EI}\dfrac{w(L-a)a}{6L}(L^2-a^2)}$

problem:-

Find the maximum deflection and maximum slope for the bar as shown fig below. Also find the deflection of a point 3.2 m from A. Take $EI = 15\times10^7$ KN/mm²



-: TORSION :-

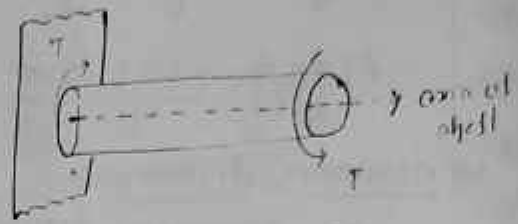
A member is said to be in torsion when it is subjected to moment about its axis. The effect of torsional moment is to twist the shaft and hence torsional moment is called as twisting.

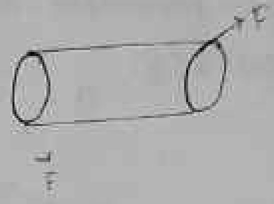
In engineering practice many members are subjected to torsion.

eg → shafts transmitting power from engine to rear axle

→ Shaft of gearbox

→ Electric motor shaft

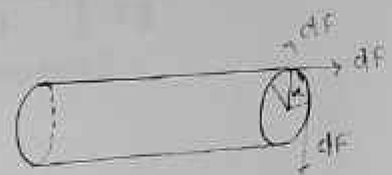




pure torsion:-

A member is said to be in pure torsion when its cross-section is subjected to only one torsional moment and not accompanied by axial forces or bending moment.

Now consider the section of a shaft under pure torsion. Internal forces develops in order to counter act this torsional moment. So at any element the force 'dF' developed is in the direction normal to radial direction.



The force is obviously shearing force and then the element is in pure shear. If 'dA' is the area of the element at distance r from the axis of rotation of shaft then,

$$dF = \tau \times dA \quad (\tau = \text{shear stress})$$

$$dT = dF \times r$$

Assumptions in theory of pure Torsion:-

→ The material is homogeneous and isotropic.

→ The stresses are within the elastic limit i.e. shear stress is directly proportional to shear strain.

→ Cross section which are plane before twisting remain plane after twisting.

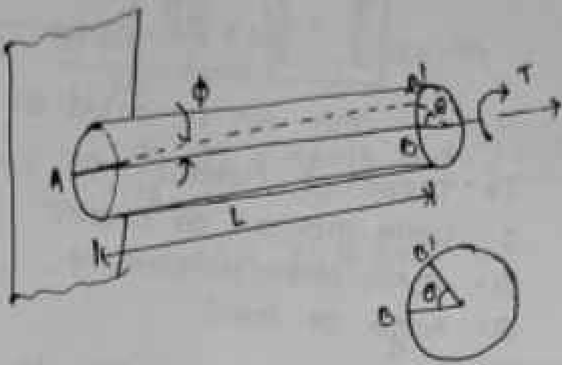
→ Radial lines remain radial even after applying torsional moment.

→ The twist along the shaft is uniform.

Derivation of Torsional Equation:-

Consider a shaft of length 'L', Radius 'R' fixed at one end and subjected to a torque T at the other end.

Let 'O' be the centre pt of circular cross section on 'B' be a point on surface. 'AB' is the line on shaft parallel to the axis of shaft.



Due to application of torque pt. B is shifted to B'. If 'ϕ' is the shear strain ($\angle BOB'$) and 'θ' be the angle of twist in length L. then,

$$R\theta = BB' = L\phi$$

If 'τ_s' is the shear stress and 'G' modulus of rigidity then

$$\phi = \tau_s/G$$

$$\therefore R\theta = L\phi = L\frac{\tau_s}{G} \qquad [\tau_s = \text{shear stress at surface}]$$

$$\boxed{\frac{\tau_s}{R} = \frac{G\theta}{L}} \quad \text{—— (1)}$$

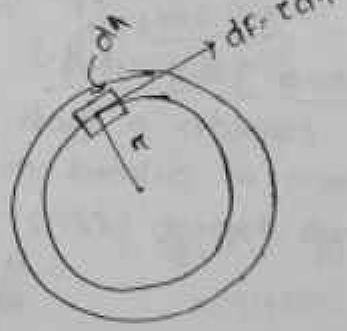
Similarly if pt B is considered at any point in between radius at any distance 'r' from centre instead of surface it can be shown that

$$\boxed{\frac{\tau}{r} = \frac{G\theta}{L}} \quad \text{—— (2)} \qquad [\tau = \text{shear stress at any radius}]$$

Now consider the torsional resistance developed by an area 'ΔA' at distance 'r' from centre

If 'τ' is the shear ~~force~~ stress then force on ΔA

$$dF = \tau dA$$



Resisting torsional moment

$$dT = dF \times r$$
$$= \tau \times dA \times r$$

From eqn (1) & (2)

$$\frac{\tau_s}{R} = \frac{\tau}{r} \Rightarrow \tau = \tau_s\frac{r}{R}$$

Now $dT = \tau_s \cdot \frac{r}{R} \cdot dA\, r = \frac{\tau_s}{R} r^2 dA$

Total torque $T = \int dt = \int \frac{\tau_s}{R} r^2 dA = \frac{\tau_s}{R}\int r^2 dA = \frac{\tau_s}{R} J$

Here $\int r^2 dA = J$ (polar moment of Inertia)

So, $\frac{T}{J} = \frac{\tau_s}{R}$ —— ③

from eqⁿ ① & ③

$$\boxed{\frac{T}{J} = \frac{\tau_s}{R} = \frac{G\theta}{L}} \quad \text{—— ④}$$

This eqⁿ is called as <u>Torsional equation.</u>

T = Torsional Moment
J = polar moment of inertia
τ_s = shear stress on element
θ = Angle of twist
R = Distance of element from centre of shaft
G = Modulus of rigidity
L = Length of shaft

This equation is similar to the eqⁿ of bending

$$\frac{M}{I} = \frac{\sigma_b}{y} = \frac{E}{R}$$

<u>polar Modulus:</u>

From Torsion equation, $\frac{T}{J} = \frac{\tau_s}{R}$

$$T = \frac{J}{R}\tau_s \Rightarrow \boxed{T = Z_p \tau_s} \quad \text{—— ⑤}$$

Z_p : polar Modulus of section

(i) Circular section of diameter 'd' -

$$J = \pi/32\, d^4 \quad , R = d/2$$

$$Z_p = J/R = \pi/32\, d^4 / d/2 = \boxed{\pi/16\, d^3}$$

(ii) Hollow circular section with external diameter (D) & internal diameter(d)

$$J = \pi/32\,(D^4 - d^4) \quad , R = D/2$$

$$Z_p = J/R = \pi/32\,(D^4 - d^4)/D/2 = \boxed{\frac{\pi}{16}\frac{D^4 - d^4}{D}}$$

<u>power Transmitted:-</u>

Consider a shaft subjected to a torque 'T' and rotating at 'N' rpm. power is defined as a rate of doing work. Taking second as a unit angle through which torque moves

$$= \frac{N}{60} \times 2\pi = \frac{2\pi N}{60}$$

power = work done per second

$$= T \times 2\pi N/60$$

$$\boxed{P = \frac{2\pi N T}{60}} \quad \text{—— ⑥}$$

If 'T' is taken as N-m, then unit of power is N-m/s i.e <u>watt</u>

$\tau_{max} = 1.3\, T_{mean}$

$T_{mean} = T = \frac{60P}{2\pi N} = \frac{80 \times 75 \times 10^3}{2 \times \pi \times 200} = 3580.98$ N-m
$= 10^3 \times 3580.98$ N-mm.

we know $\frac{T}{J} = \frac{\tau}{R} \Rightarrow \frac{J}{R} = \frac{T}{\tau} = \frac{3580.98 \times 10^3}{70} = 51156.94$

$\Rightarrow \frac{\pi/32\, d^4}{d/2} = 51156.94 \Rightarrow \frac{\pi}{16} d^3 = 51156.94 \Rightarrow d = \sqrt[3]{\frac{51156.94 \times 16}{\pi}}$

$\Rightarrow \boxed{d = 63.86 \text{ mm}}$

problem:-

The working condition to be satisfied by a shaft transmitting power are (i) the shaft must not twist more than 1° in a length of 15 times diameter (ii) the shear stress must not exceed 80 MN/m². What the actual working stress and diameter of the shaft should taken to transmit 736 KW at 200 rpm. Take shear modulus as 80 GN/m².

Sol^n

$\theta = 1° = \pi/180$ rad

$L = 15d$

$\tau_{max} = 80$ MN/m² $= 80$ N/mm²

$P = 736$ KW $= 736 \times 10^3$ watt $\qquad G = 80 \times 10^9$ N/m² $= 80 \times 10^3$ N/mm²

$N = 200$ rpm

$P = \frac{2\pi NT}{60} \Rightarrow T = \frac{60P}{2\pi N} = \frac{60 \times 736 \times 10^3}{2 \times \pi \times 200} = 35141.41$ N-m $= 35141.41 \times 10^3$ N-mm

~~J = π/16 d⁴~~

$\frac{T}{J} = \frac{G\theta}{L} \Rightarrow \theta = \frac{TL}{GJ} \Rightarrow \frac{35141.41 \times 10^3 \times 15d}{80 \times 10^3 \times \pi/32\, d^4} = \frac{67115.14}{d^3}$

$\Rightarrow \frac{67115.14}{d^3} = \pi/180 \Rightarrow d^3 = \frac{67115.14 \times 180}{\pi} = 3845414.79$

$\Rightarrow \boxed{d = 156.66 \text{ mm}}$

corresponding stress developed:

$J = \pi/32\, d^4 = \pi/32\, (156.66)^4$
$= 59145832.88$ mm⁴

$\frac{T}{J} = \frac{\tau}{R} \Rightarrow \frac{35141.41 \times 10^3}{59145832.88} = \frac{\tau}{\frac{156.66}{2}}$

$\Rightarrow \boxed{\tau = 46.53 \text{ N/mm}^2}$

As τ_{max} is 80 N/mm²
$\tau < \tau_{max}$ so the shaft is safe.

Torsional Rigidity / stiffness of shafts:-

we know $\frac{T}{J} = \frac{G\theta}{L}$

$\Rightarrow \boxed{T = \frac{GJ\theta}{L}}$ ——— (7)

problem:-

Calculate the maximum intensity of shear stress induced and the angle of twist produced in degrees of a solid shaft 120 mm diameter, 12 m long, transmitting 120 KW at 150 rpm. Take G = 82 kN/mm²

Soln

Given data,

Dia of shaft (d) = 120 mm

length of shaft (L) = 12m = 12000 mm

power transmitted (P) = 120 KW = 120×10^3 watt.

RPM = 150

$G = 82 \text{ kN/mm}^2 = 82000 \text{ N/mm}^2$

→ Torque transmitted (T)

$$P = \frac{2\pi NT}{60} \Rightarrow T = \frac{60 \times P}{2\pi N} = \frac{60 \times 120 \times 10^3}{2 \times \pi \times 150} = 7639.43 \text{ N-m}$$
$$= 7639.43 \times 10^3 \text{ N-mm}$$

→ polar moment of inertia (J) = $\frac{\pi}{32} d^4 = \frac{\pi}{32} \times (120)^4 = 20357520.4 \text{ mm}^4$

→ Intensity of shear stress (τ_s) =

$$\frac{T}{J} = \frac{\tau_s}{R} \Rightarrow \tau_s = \frac{T}{J} R = \frac{7639.43 \times 10^3}{20357520.4} \times 60 = \boxed{22.51 \text{ N/mm}^2}$$

→ Angle of twist (θ) =

$$\frac{T}{J} = \frac{G\theta}{L} \Rightarrow \theta = \frac{TL}{GJ} = \frac{7639.43 \times 10^3 \times 12000}{82000 \times 20357520.4} = 0.0549 \text{ rad}$$

$$= 0.0549 \times \frac{180}{\pi} = \boxed{3.14^\circ}$$

problem:-

A solid steel shaft has to transmit 75 KW at 200 rpm. Taking allowable shear stress as 70 N/mm², Find the allowable diameter to the shaft if maximum torque transmitted on each revolution exceeds the mean by 30%.

Soln

$P = 75 \text{ KW} = 75000 \text{ watt}$

$N = 200 \text{ rpm}$

$\tau = 70 \text{ N/mm}^2$

problem:-

prove that a hollow shaft is stronger and stiffer than the solid shaft of the same material, length and ~~d~~ ~~width~~ weight

Sol^n Let, d = Diameter of solid shaft

d_1 = Diameter of inner side of hollow shaft

d_2 = Outer diameter of hollow shaft.

Two shafts have equal weight and length so,

$$\pi/_4\, d^2 \rho L = \frac{\pi}{4}(d_2^2 - d_1^2)\rho L \quad \text{—①} \quad [\rho = \text{unit weight}, \; L = \text{Length}$$

T_s : Resisting torque of solid shaft

T_h : Resisting torque of hollow shaft

$$T_s = \frac{\tau}{R} J_s = \frac{\tau}{d/2}\, \pi/_{32}\, d^4 = \pi/_{16}\, \tau d^3$$

$$T_h = \frac{\tau}{R_2} J_h = \frac{\tau}{d_2/2}\, \pi/_{32}(d_2^4 - d_1^4) = \frac{\pi}{16}\frac{(d_2^4 - d_1^4)}{d_2}\tau$$

$$\frac{T_h}{T_s} = \frac{\cancel{\pi/16}\,(d_2^4 - d_1^4)\cancel{\tau}}{\cancel{\pi/16}\,\cancel{\tau}\, d^3 d_2} = \frac{d_2^4 - d_1^4}{d^3 d_2} \quad \text{—②}$$

From eq^n ① $d^2 = d_2^2 - d_1^2$

putting it eq^n ②

$$\frac{T_h}{T_s} = \frac{d_2^4 - d_1^4}{d^2 d \cdot d_2} = \frac{d_2^4 - d_1^4}{(d_2^2 - d_1^2) d \cdot d_2} = \frac{(d_2^2 + d_1^2)\cancel{(d_2^2 - d_1^2)}}{\cancel{(d_2^2 - d_1^2)}\, d \cdot d_2} = \frac{d_2^2 + d_1^2}{d \cdot d_2}$$

$$= \frac{d_2^2 + d_1^2}{\sqrt{d_2^2 - d_1^2} \cdot d_2} = \frac{\cancel{d_2^2}\,(1 + (d_1/d_2)^2)}{\cancel{d_2}\sqrt{1 - \left(\frac{d_1}{d_2}\right)^2} \cdot \cancel{d_2}} = \frac{1 + \left(\frac{d_1}{d_2}\right)^2}{\sqrt{1 - \left(\frac{d_1}{d_2}\right)^2}} \quad \text{—③}$$

From the eq^n ③ we get that numerator is > 1 and denominator < 1

so $T_h > T_s$

Hollow shaft is stronger than solid shaft

Stiffer:-

Stiffness may be defined as the torque required to produce unit rotation in unit length. So

$$\frac{K}{J} = \frac{G \times 1}{1} \Rightarrow K = GJ \quad \text{—④}$$

K_s : Stiffness of solid shaft

K_h : Stiffness of hollow shaft

$$K_s = GJ_s = G\frac{\pi}{32}d^4$$

$$K_h = GJ_h = G\frac{\pi}{32}(d_2^4 - d_1^4)$$

problem:-

During the test on a sample of steel bar 25 mm in diameter, it was found that the pull of 50 kN produces an extension of 0.095 mm on a length of 200 mm and a torque 200 N·m produces an angular twist of 0.9° on a length of 250 mm. Find poisson's ratio of steel.

Sol^n $d = 25$ mm, $P = 50 \times 10^3$ N, $\Delta l = 0.095$ mm, $L = 200$ mm.

$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times (25)^2 = 490.87 \text{ mm}^2$

$E = \frac{\sigma}{e} = \frac{P/A}{\Delta l / l} = \frac{PL}{A \Delta l} = \frac{50 \times 10^3 \times 200}{490.87 \times 0.095} = 214442.02 \text{ N/mm}^2$

$T = 200$ N·m $= 200 \times 10^3$ N·mm, $\theta = 0.9° = 0.9 \times \frac{\pi}{180} = 0.0157$ rad.

$L = 250$ mm., $J = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times (25)^4 = 38349.51 \text{ mm}^4$

$\frac{T}{J} = \frac{G\theta}{L} \Rightarrow G = \frac{TL}{J\theta} = \frac{200 \times 10^3 \times 250}{38349.51 \times 0.0157} = 83044.43 \text{ N/mm}^2$

we know the relation

$E = 2G(1+\mu) \Rightarrow 1+\mu = \frac{E}{2G} \Rightarrow \mu = \frac{E}{2G} - 1$

$\mu = \frac{214442.02}{2 \times 83044.43} - 1 = \boxed{0.291}$

problem:-

A hollow circular shaft of 7 m length and inner and outer diameters of 75 mm and 125 mm is subjected to a torque of 15 kN-m. If $G = 80$ GPa Determine the maximum shear strain produced and total angle of twist.

Sol^n $L = 7$ m $= 7000$ mm

$d = 75$ mm, $D = 125$ mm

$T = 15$ kN-m $= 15 \times 1000$ N-m $= 15000 \times 10^3$ N-mm

$G = 80$ GPa $= 80 \times 10^9$ N/m^2 $= 80 \times 10^3$ N/mm^2

$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}\left[(125)^4 - (75)^4\right]$

$= 20862138.72 \text{ mm}^4$

$\frac{T}{J} = \frac{G\theta}{L}$

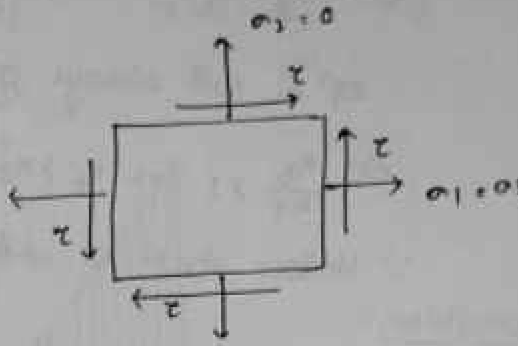
$\Rightarrow \frac{15000 \times 10^3}{20862138.72} = \frac{80 \times 10^3 \times \theta}{7000}$

$\Rightarrow \theta = 0.06 \text{ rad} = \boxed{3.6°}$

$\frac{T}{J} = \frac{\tau}{R} \Rightarrow \frac{15000 \times 10^3}{20862138.37} = \frac{\tau}{125/2} \Rightarrow \boxed{\tau = 44.93 \text{ N/mm}^2}$

Shafts subjected to combined bending & Twisting:

Some times shafts subjected to both bending and twisting moment due to its self weight and rotation. Then we have to find out equivalent bending moment and twisting moment



$$\sigma_b = \frac{M}{I} y = \frac{M}{\frac{\pi}{64} d^4} \times \frac{d}{2} = \frac{32M}{\pi d^3}$$

$$\tau = \frac{T}{J} R = \frac{T}{\frac{\pi}{32} d^4} \times \frac{d}{2} = \frac{16T}{\pi d^3}$$

Then principal stresses are

$$\sigma_1 = \frac{\sigma_b + 0}{2} \pm \sqrt{\left(\frac{\sigma_b - 0}{2}\right)^2 + \tau^2}$$

$$= \frac{32M}{2\times\pi d^3} \pm \sqrt{\left(\frac{32M}{2\times\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3} \pm \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$= \frac{32}{\pi d^3} \frac{1}{2}\left[M \pm \sqrt{M^2 + T^2}\right]$$

$$\Rightarrow \frac{\pi}{32} \sigma_1 d^3 = \frac{1}{2}\left[M \pm \sqrt{M^2+T^2}\right] \Rightarrow \boxed{M_e = \frac{1}{2}\left[M \pm \sqrt{M^2+T^2}\right]}$$

M_e: Equivalent bending moment.

Maximum shear stress

$$\tau_{max} = \sqrt{\left(\frac{\sigma_b - 0}{2}\right)^2 + \tau^2}$$

$$= \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$= \sqrt{\left(\frac{32M}{\pi d^3 2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$

$$\Rightarrow \frac{\pi}{16} \tau_{max} d^3 = \sqrt{M^2+T^2}$$

$$\Rightarrow \boxed{T_e = \sqrt{M^2+T^2}}$$

T_e: Equivalent turning moment

$$\frac{K_h}{K_s} = \frac{d_2^4 - d_1^4}{d^4} \quad \text{—— (5)}$$

Substituting the value of eqⁿ (1) in eqⁿ (5)

$$\frac{K_h}{K_s} = \frac{d_2^4 - d_1^4}{(d_2^2 - d_1^2)^2} = \frac{(d_2^2 + d_1^2)(d_2^2 - d_1^2)}{(d_2^2 - d_1^2)^2} = \frac{d_2^2 + d_1^2}{d_2^2 - d_1^2} \quad \text{— (6)}$$

eqⁿ (6) will always greater than 1.

So $\frac{K_h}{K_s} > 1 \Rightarrow K_h > K_s$

∴ Hollow shaft is stiffer than solid shaft

Problem:-

A solid shaft transmits 250 kW at 100 rpm. If the shear stress does not to exceed 70 N/mm², what should be the diameter of the shaft? If the shaft is to be replaced by a hollow one whose internal diameter = 0.6 times than outer diameter.

Solⁿ

$P = 250\,kW = 250 \times 10^3$ watt, $N = 100$ rpm, $\tau = 70\,N/mm^2$

$$T = \frac{60P}{2\pi N} = \frac{60 \times 250 \times 10^3}{2 \times \pi \times 100} = 23873.24 \times 10^3 \text{ N-mm}$$

For solid shaft:-

$$\frac{T}{J} = \frac{\tau}{r} \Rightarrow T = \frac{\pi}{16} \tau d^3 \Rightarrow d = \sqrt[3]{\frac{16T}{\pi\tau}} = \sqrt[3]{\frac{16 \times 23873.24 \times 10^3}{\pi \times 70}} = \boxed{120.20\text{ mm}}$$

For hollow shaft:-

$d = 0.6D$

$$\frac{T}{J} = \frac{\tau}{R} \Rightarrow T = \tau \times \frac{J}{R} = \tau \times \frac{\pi}{16}\,\frac{D^4 - d^4}{D} = 70 \times \frac{\pi}{16} \times \frac{D^4 - (0.6D)^4}{D} = 11.96\,D^3$$

$\Rightarrow 23873.24 \times 10^3 = 11.96\,D^3 \Rightarrow D = 125.90$ mm $\Rightarrow d = 75.54$ mm.

Cross section of solid shaft $= \frac{\pi}{4} d^2 = \frac{\pi}{4}(120.20)^2 = 11347.46\ mm^2$

Cross section of hollow shaft $= \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[(125.90)^2 - (75.54)^2\right]$

$= 7967.48\ mm^2$

%age saving in weight $= \frac{\text{weight solid} - \text{weight hollow}}{\text{weight solid}}$

$$= \frac{(11347.46 - 7967.48)\,\rho L}{11347.46\,\rho L} = 0.2978$$

$= \boxed{29.78\,\%}$

Strain Energy in Torsion:-

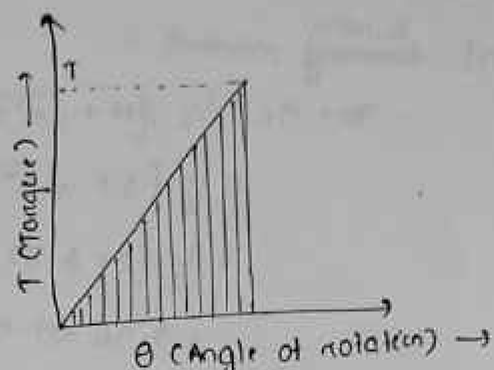
When Torque T is applied to a shaft, it gets twisted by an angle θ. Then the twisting moment so developed is stored in the shaft in terms of strain energy. When T is gradually increased angle θ also gradually increases. If we will plot a graph by taking θ in x-axis and T in y-axis

Strain Energy (U) = work done

$= $ Area under curve

$= \frac{1}{2} T\theta$

$= \frac{1}{2} T \left(\frac{TL}{GJ}\right) = \frac{1}{2} \frac{T^2 L}{GJ}$



→ Solid shaft

$J = \frac{\pi}{32} d^4$

$T = \frac{\tau}{R} J$

$U = \frac{1}{2}\frac{T^2 L}{GJ} = \frac{1}{2} \frac{\tau^2}{(d/2)^2} J^2 \cdot \frac{L}{GJ} = \frac{1}{2}\frac{\tau^2}{d^2} \times 4 \times \left(\frac{\pi}{32} d^4\right)^2 \cdot \frac{L}{G \cdot (\pi/32 \, d^4)}$

$= \frac{1}{2} \frac{\tau^2}{d^2} \times 4 \times \frac{\pi}{32} d^4 \times \frac{L}{G} \Rightarrow \frac{\pi}{16} \tau^2 d^2 \frac{L}{G} = \frac{\pi}{4} d^2 L \times \frac{\tau^2}{4G}$

$\Rightarrow \frac{\tau^2}{4G} \times$ volume

$\therefore$ strain energy per unit volume $= \tau^2/4G$

→ Hollow shaft

$J = \pi/32 (D^4 - d^4)$

$U = \frac{1}{2}\frac{T^2 L}{GJ} = \frac{1}{2} \frac{\tau^2}{(D/2)^2} J^2 \cdot \frac{L}{GJ} = 2\frac{\tau^2}{D^2} J^2 \frac{L}{GJ} = \frac{2\tau^2 J L}{D^2 G}$

$= 2\tau^2 \frac{\pi}{32} \frac{(D^4 - d^4) L}{D^2 G} = \frac{\pi}{16} \tau^2 \frac{(D^2+d^2)(D^2-d^2)}{D^2 G} L$

$= \frac{\pi}{4}(D^2 - d^2) \cdot \frac{\tau^2}{4} \frac{D^2+d^2}{D^2 G} L = \text{volume} \times \frac{\tau^2 (D^2+d^2)}{4D^2 G}$

$\therefore$ strain energy per unit volume $= \frac{\tau^2 (D^2 + d^2)}{4D^2 G}$

Problem:-

A circular shaft 2m long is required to transmit 1000 kw at 300 rpm. If the outer diameter of shaft is 150 mm and inner diameter is 120 mm, find the maximum shear stress and strain energy stored in shaft. Take G=80 kN/mm² roblem:-

At a certain cross section bending moment and twisting moment are recorded as 3KN-m and 6KN-m. Find the diameter to design the shaft if $\sigma_b = 100 N/mm^2$, $\tau = 50 N/mm^2$

sol^n

$M = 3 KN\text{-}m = 3000 N\text{-}m$

$T = 6 KN\text{-}m = 6000 N\text{-}m$

Equivalent ~~twisting~~ Bending moment

$$M_e = \frac{1}{2}\left[M + \sqrt{M^2+T^2}\right]$$

$$= \frac{1}{2}\left[3 + \sqrt{3^2+6^2}\right]$$

$$= \frac{1}{2}\left[3 + 6.92\right]$$

$$= 4.96 \ KN\text{-}m$$

Equivalent twisting moment

$$T_e = \sqrt{M^2+T^2}$$

$$= \sqrt{3^2+6^2}$$

$$= 6.92 \ KN\text{-}m$$

$T_e > M_e$ so design will be done based on T_e

$$T_e = 6.92 \ KN\text{-}m = 6.92 \times 10^3 \times 10^3 \ N\text{-}mm$$

$$T_e = \frac{\pi}{16} \tau d^3$$

$$\Rightarrow d = \sqrt[3]{\frac{16 T_e}{\pi \tau}} = \sqrt[3]{\frac{16 \times 6.92 \times 10^6}{\pi \times 50}}$$

$$= \boxed{88.99 mm}$$

Problem:-

A close coiled helical spring is made with 12mm diameter wire and is having mean diameter of 150 mm and 10 complete turns. The modulus of rigidity of spring material is 80 kN/mm². When a load of 450 kN is applied find → Maximum shear stress

→ Strain energy stored

→ Deflection produced

→ Stiffness of spring.

Given, $P = 1000\ kW = 1000 \times 10^3$ watt, $N = 300$ rpm

$D = 150$ mm, $d = 120$ mm, $G = 80\ kN/mm^2 = 80000\ N/mm^2$, $L = 2000$ mm

$$T = \frac{60P}{2\pi N} = \frac{60 \times 1000 \times 10^3}{2 \times \pi \times 300} = 31830.98\ N\text{-}m = 31830.98 \times 10^3\ N\text{-}mm$$

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}((150)^4 - (120)^4) = 29343457.13\ mm^4$$

$$\tau = \frac{T}{J}R = \frac{31830.98 \times 10^3}{29343457.13} \times 75 = \boxed{81.35\ N/mm^2}$$

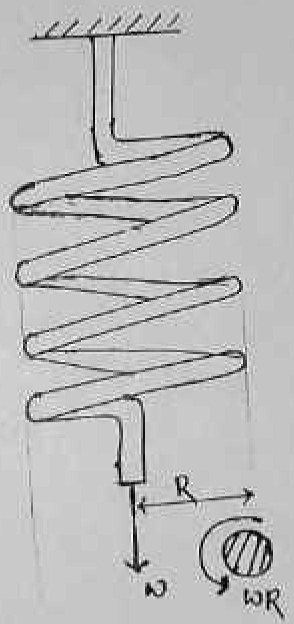
Strain energy stored :-

$$U = \frac{\tau^2(D^2+d^2)}{4D^2G} \times \frac{\pi}{4}(D^2-d^2) \times L$$

$$= \frac{(81.35)^2(150^2+120^2)}{4 \times 150^2 \times 80 \times 1000} \times \frac{\pi}{4}(150^2 - 120^2) \times 2000$$

$$= 1726131.47\ N\text{-}mm = 1726.131\ N\text{-}m = \boxed{1726.131\ \text{Joule}}$$

Close coiled Hellical Spring :-



The fig. shown a close coiled hellical spring where,

R = Radius of spring

w = Axial load, n = Total no. of coils.

d = Diameter of spring wire.

So, Entire length of spring wire $= 2\pi R \times n$

Torsional moment $= T = WR$

From torsion formula

$$\frac{T}{J} = \frac{\tau}{r} \Rightarrow T = \frac{\pi}{16}\tau d^3$$

$$\Rightarrow \tau = \frac{16T}{\pi d^3} = \frac{16WR}{\pi d^3}$$

Strain energy in spring $= \frac{\tau^2}{4G} \times$ volume

$$= \left(\frac{16WR}{\pi d^3}\right)^2 \times \frac{1}{4G} \times 2\pi R n \times \frac{\pi}{4}d^2$$

$$= \frac{32W^2R^3n}{Gd^4} \quad \text{——— (1)}$$

If δ is deflection, work done $= \frac{1}{2}W\delta$ ——— (2)

eq^n (1) = eq^n (2)

$$\frac{1}{2}W\delta = \frac{32W^2R^3n}{Gd^4} \Rightarrow \delta = \frac{64WR^3n}{Gd^4} \quad \text{——— (3)}$$

Stiffness of spring $= \frac{\text{Load}}{\text{Deflection}} = \frac{W}{\delta} = \frac{W}{\frac{64WR^3n}{Gd^4}} = \frac{Gd^4}{64R^3n}$ ——— (4)